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DIFFUSION IN A SOLID CYLINDER PART II: DIFFUSION DEPTH

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ABSTRACT

When advancing diffusion model was applied in deriving the solution of concentration of diffusing substance in a solid cylinder medium, calculating diffusion depth numerically became possible. However, the numerical iteration makes the process of curve fitting the data of the concentration of diffusing substance in the medium very difficult, which is a critical process of deriving the diffusion related parameters. To prevent the numerical iteration, this study presents a process named Taylor series approaching illation. A polynomial is derived for locating the diffusing front without numerical iteration. Numerical fitting the data of the average concentration of the diffusion substance in the cylinder medium to derive the diffusion related parameter becomes straightforward.

Keywords: The advancing model, diffusion.

1. INTRODUCTION

Fick's diffusion law has been used in deriving the concentration of diffusing substance in a semi-infinite medium, a solid cylinder medium and many other cases. Based on Fick's diffusion law, Carslaw and Jaeger applied Laplace transform to solve the solution of concentration of diffusing substance in a case that the diffusing substance penetrates the surface of a semi-infinite medium and is transported inward (Crank 1975). In the same year, they adopted a technique developed by Neumann for heat conduction to derive the solution of the concentration of diffusing substance with an advancing boundary in a semi-infinite medium, named the advancing model, that makes calculating the diffusion depth possible (Crank 1975; Carslaw and Jarger 1986; Chang *et al*. 2008; Wang 2010; Wang *et al*. 2011).

Carslaw and Jaeger also derived the solution for a solid cylinder medium in terms of Bessel function without the advancing model (Crank 1975). The complexity of cylinder coordinate system forms a great hurdle that obstacles researchers from deriving the solution with the advancing model for the last half a century until Tsai, Lin, Wang and Lee (Tsai *et al*. 2015).

Tsai, Lin, Wang and Lee successfully adopted the advancing model in cylinder coordinate system by correlating the local diffusion around the surface of a cylinder medium to that of a semi-infinite medium. Calculating diffusion depth became possible for a cylinder medium. Different from the case for a semi-infinite medium, the result does not yield a succinct function for diffusion depth. The diffusion depth must be calculated tediously by numerical process, which is used to calculate the average concentration of diffusion substance in the medium to fit the experimental data in the process of least square curve fitting for deriving the diffusion related properties.

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This study presents a process based on illation of Taylor series to derive polynomials to calculate critical diffusion duration and diffusion depth analytically. The polynomials are smooth and derivable and will be used to simplify tremendously the tedious process of fitting data of average concentration of diffusing substance in a cylinder specimen of medium for characterizing diffusing properties.

The solutions can also be used in calculating advancing speed of the diffusion front in its way toward the center of the medium. One of the discoveries from the result is that the advancing speed of the diffusion front is infinite at the very beginning and at the center of the medium, which is novel and will be important in the study of the efficiency of a reverse osmosis devise.

2. THEORETICAL BACKGROUND

The governing Eq. of Fick's second diffusion law (Crank 1975) is

$$\frac{\partial c}{\partial t} = D\nabla^2 c \tag{1}$$

where, c is the concentration of diffusing substance in a medium defined as mass of diffusing substance in unit volume of medium, t is diffusion duration, and D is diffusivity which is a positive constant. In three-dimensional Cartesian coordinate system, Eq. (1) becomes

$$\frac{\partial c}{\partial t} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right) \tag{2}$$

where, x, y and z are the three coordinates. In an one-dimensional case, Eq. (2) becomes

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} \tag{3}$$

In a semi-infinite case, the medium exists only in the region where $x \geq 0$ and is free of diffusing substance initially. The diffusing substance existing in the region where $x < 0$ will penetrate the surface of the medium and be transported inwardly following Fick's diffusion law in the medium (Fig. 1). This is a typical one-dimensional diffusion case, where c is a function of t and x, $c(t; x)$. The assumed boundary condition is that when the diffusing substance comes into contact with the medium at the boundary, the boundary becomes saturated instantly and remains so throughout the process of diffusion.

$$c(t;0) = c_\infty \tag{4}$$

where c_∞ is the saturated c, and the initial condition is that

$$c(0;x) = 0 \tag{5}$$

When the advancing model is not considered, the solution of $c(t; x)$ of Eq. (3) considering boundary condition, Eq. (4), and initial condition, Eq.(5), is

$$\begin{aligned} c(t;x) &= c_\infty \left[1 - erf\left(\frac{x}{2\sqrt{Dt}}\right)\right] \\ &= c_\infty \left[1 - \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{x}{2\sqrt{Dt}}\right)^{2n+1}}{n!(2n+1)}\right] \end{aligned} \tag{6}$$

where *erf* stands for error function [1]. When the advancing model is considered, Eq. (6) becomes

$$\frac{c(t;x)}{c_\infty} = \begin{cases} 1 - \dfrac{\sum_{n=0}^{\infty} (-1)^n \dfrac{\left(\frac{x}{2\sqrt{Dt}}\right)^{2n+1}}{n!(2n+1)}}{C}, \\ C = \sum_{n=0}^{\infty} (-1)^n \dfrac{\nu^{2n+1}}{n!(2n+1)}, \quad 0 \le x \le 2\nu\sqrt{Dt} \\ 0, \quad 2\nu\sqrt{Dt} \le x \end{cases} \tag{7}$$

where C is correction factor of the advancing model, and is a constant named Neumann's constant after Neumann F. (Crank 1975).

Eq. (7) implies a clear result that the first equation on the right hand side vanishes at

$$\frac{x}{2\sqrt{Dt}} = \nu \quad or \quad x = 2\nu\sqrt{Dt} \tag{8}$$

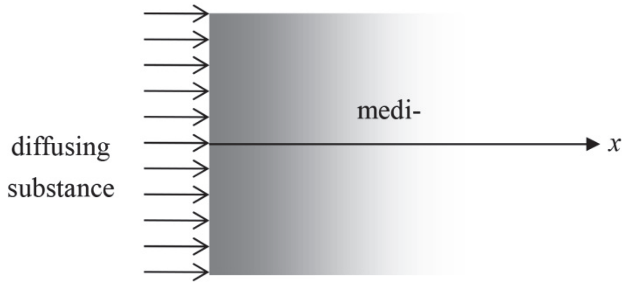

Fig. 1 Diffusion in a semi-infinite medium

and is forced to be zero at any location deeper than $2\nu\sqrt{Dt}$ which is obviously the diffusion front that does not exist in conventional solution of c, Eq. (6).

For cylinder coordinate system, Eq. (1) becomes

$$\frac{\partial c}{\partial t} = D\left(\frac{\partial^2 c}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r} + \frac{1}{r^2}\frac{\partial^2 c}{\partial \theta^2} + \frac{\partial^2 c}{\partial z^2}\right) \tag{9}$$

where r and θ are the polar coordinates, and z is the axial coordinate. When the case is axially symmetrical diffusion in an infinitely long solid cylinder with radius r_a, c will be dependent on t and r only, $c(t; r)$, and Eq. (9) becomes

$$\frac{\partial c}{\partial t} = D\left(\frac{\partial^2 c}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r}\right) \tag{10}$$

For the case that the initial concentration of the diffusing substance in the cylinder is zero,

$$c(0; r) = 0 \tag{11}$$

and the diffusing substance existing in the surrounding environment penetrates the surface when the diffusion process starts and travels toward the center of the cylinder (Fig. 2). Same as that in previous case, the assumed boundary condition is that when the diffusing substance comes into contact with the boundary of the medium, the boundary becomes saturated instantly and remains so throughout the process of diffusion.

$$c(t; ra) = c_\infty \tag{12}$$

When the advancing model is not considered, the solution of $c(t; r)$ of Eq. (10) considering the boundary condition, Eq. (12), and the initial condition, Eq. (11), is

$$\frac{c(t;r)}{c_\infty} = 1 - \sum_{m=1}^{\infty}\left[\frac{2}{\alpha_m J_1(\alpha_m)} J_0\left(\alpha_m \frac{r}{r_a}\right) e^{-\left(\alpha_m \frac{\sqrt{Dt}}{r_a}\right)^2}\right] \tag{13}$$

where J_0 and J_1 are first kind of Bessel function of order zero and order one (Crank 1975; Wang *et al*. 2011; Tsai *et al*. 2015) α_m is the m-th zero of J_0 listed in Table 1, which are calculated numerically by letting $J_0(\alpha_m) = 0$ (Tsai *et al*. 2015; Abramowitz and Stegun 1972).

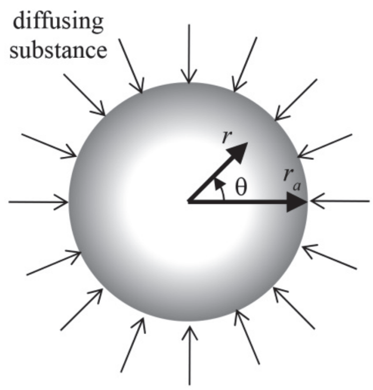

Fig. 2 Diffusion in a cylinder medium

Table 1 Zeros of J_0

m	α_m	m	α_m	m	α_m
1	2.4048255604	34	106.0299309171	67	209.7019057049
2	5.5200781056	35	109.1714896504	68	212.8434895606
3	8.6537279135	36	112.3130502811	69	215.9850736721
4	11.7915344396	37	115.4546126543	70	219.1266580286
5	14.9309177091	38	118.5961766315	71	222.2682426197
6	18.0710639685	39	121.7377420886	72	225.4098274355
7	21.2116366305	40	124.8793089138	73	228.5514124667
8	24.3524715314	41	128.0208770066	74	231.6929977046
9	27.4934791327	42	131.1624462758	75	234.8345831410
10	30.6346064690	43	134.3040166389	76	237.9761687679
11	33.7758202142	44	137.4455880209	77	241.1177545779
12	36.9170983543	45	140.5871603535	78	244.2593405639
13	40.0584257652	46	143.7287335743	79	247.4009267193
14	43.1997917138	47	146.8703076264	80	250.5425130376
15	46.3411883723	48	150.0118824576	81	253.6840995128
16	49.4826098980	49	153.1534580198	82	256.8256861392
17	52.6240518417	50	156.2950342691	83	259.9672729112
18	55.7655107556	51	159.4366111649	84	263.1088598237
19	58.9069839267	52	162.5781886695	85	266.2504468717
20	62.0484691908	53	165.7197667486	86	269.3920340504
21	65.1899648008	54	168.8613453698	87	272.5336213553
22	68.3314693305	55	172.0029245037	88	275.6752087821
23	71.4729816042	56	175.1445041225	89	278.8167963268
24	74.6145006443	57	178.2860842007	90	281.9583839852
25	74.6145006443	58	181.4276647143	91	285.0999717538
26	80.8975558717	59	184.5692456412	92	288.2415596288
27	84.0390907775	60	187.7108269607	93	291.3831476069
28	87.1806298442	61	190.8524086532	94	294.5247356847
29	90.3221726378	62	193.9939907007	95	297.6663238591
30	93.4637187825	63	197.1355730863	96	300.8079121270
31	96.6052679516	64	200.2771557939	97	303.9495004856
32	99.7468198593	65	203.4187388088	98	307.0910889321
33	102.8883742548	66	206.5603221168	99	310.2326774638
				100	313.3742660781

When the advancing model is considered (Tsai *et al*. 2015), Eq. (13) becomes

$$\frac{c(t;r)}{c_\infty}=\begin{cases}1-\dfrac{\sum_{m=1}^{\infty}\left[\dfrac{2}{\alpha_m J_1(\alpha_m)}J_0\left(\alpha_m\dfrac{r}{r_a}\right)e^{-\left(\alpha_m\frac{\sqrt{Dt}}{r_a}\right)^2}\right]}{C}, & r\ge r_f\\ 0, \quad r\le r_f\end{cases}\tag{14}$$

for $t \le t_c$, where tc is the critical diffusion duration when the diffusion front reaches the center of the medium, rf is the location of diffusion front, and diffusion depth is r_a - r_f. When $t \ge t_c$, the medium is contaminated thoroughly, diffusion front disappears, and Eq. (14) becomes

$$\frac{c(t;r)}{c_\infty}=1-\frac{\sum_{m=1}^{\infty}\left[\dfrac{2}{\alpha_m J_1(\alpha_m)}J_0\left(\alpha_m\dfrac{r}{r_a}\right)e^{-\left(\alpha_m\frac{\sqrt{Dt}}{r_a}\right)^2}\right]}{C}\tag{15}$$

This work focus on the period when $t \le t_c$. Eq. (14) leads to

$$1-\frac{\sum_{m=1}^{\infty}\left[\dfrac{2}{\alpha_m J_1(\alpha_m)}J_0\left(\alpha_m\dfrac{r_f}{r_a}\right)e^{-\left(\alpha_m\frac{\sqrt{Dt}}{r_a}\right)^2}\right]}{C}=0\tag{16}$$

or

$$\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{\nu^{2n+1}}{n!\,(2n+1)}=C$$

$$=\sum_{m=1}^{\infty}\left[\frac{2}{\alpha_m J_1(\alpha_m)}J_0\left(\alpha_m\frac{r_f}{r_a}\right)e^{-\left(\alpha_m\frac{\sqrt{Dt}}{r_a}\right)^2}\right] \tag{17}$$

3. PROCESS OF TAYLOR SERIES APPROACHING ILLATION

3.1 Functional relationship between T_c and ν

In Eq. (17), let $R = r_f/r_a$ named normalized location of diffusion front, and $T = Dt\,/\,r_a^2$ named normalized diffusion duration, Eq. (17) becomes

$$\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{\nu^{2n+1}}{n!\,(2n+1)}=\sum_{m=1}^{\infty}\left[\frac{2}{\alpha_m J_1(\alpha_m)}J_0(\alpha_m R)e^{-\alpha_m^2 T}\right] \tag{18}$$

When the diffusion front reaches the center of the medium, $T_c = Dt_c\,/\,r_a^2$, $r_f = 0$, and $R = 0$. Eq. (18) becomes

$$\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{\nu^{2n+1}}{n!\,(2n+1)}=\sum_{m=1}^{\infty}\left[\frac{2}{\alpha_m J_1(\alpha_m)}e^{-\alpha_m^2 T_c}\right] \tag{19}$$

T_c can be calculated numerically from Eq. (19) and is sensitive to ν (Fig. 3). To derive the functional relationship between T_c and a process named Taylor series approaching illation, Tsai process in short, is developed and applied herein. Define

$$T_{c1\nu}=\frac{dT_c}{d\nu},\ T_{c2\nu}=\frac{d^2T_c}{d\nu^2},\ T_{c3\nu}=\frac{d^3T_c}{d\nu^3},\ T_{c4\nu}=\frac{d^4T_c}{d\nu^4},\ T_{c5\nu}=\frac{d^5T_c}{d\nu^5},\ T_{c6\nu}=\frac{d^6T_c}{d\nu^6},\ \cdots \tag{20}$$

Express T_c as a function of in Taylor series at $\nu = \nu_h$, where ν_h is a chosen value of ν.

$$T_c=T_{cvh}+\frac{T_{c1\nu h}}{1!}(\nu-\nu_h)+\frac{T_{c2\nu h}}{2!}(\nu-\nu_h)^2+\frac{T_{c3\nu h}}{3!}(\nu-\nu_h)^3+\frac{T_{c4\nu h}}{4!}(\nu-\nu_h)^4+\frac{T_{c5\nu h}}{5!}(\nu-\nu_h)^5+\frac{T_{c6\nu h}}{6!}(\nu-\nu_h)^6+\cdots \tag{21}$$

where T_{cvh}, $T_{c1\nu h}$, $T_{c2\nu h}$, $T_{c3\nu h}$, $T_{c4\nu h}$, $T_{c5\nu h}$ and $T_{c6\nu h}$ are T_c, T_{c1}, T_{c2}, T_{c3}, T_{c4}, T_{c5} and T_{c6} at $\nu = \nu_h$. T_{cvh} in Eq. (21) can be calculated numerically from Eq. (19) by letting $\nu = \nu_h$. Differentiate Eq. (19) with respect to ν once to derive

$$\frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{\nu^{2n}}{n!}=\frac{e^{-\nu^2}}{\sqrt{\pi}}=-T_{c1\nu}\sum_{m=1}^{\infty}\left[\frac{\alpha_m e^{-\alpha_m^2 T_c}}{J_{1m}}\right] \tag{22}$$

or

$$-T_{c1\nu}^{-1}\frac{e^{-\nu^2}}{\sqrt{\pi}}=\sum_{m=1}^{\infty}\left(\frac{\alpha_m e^{-\alpha_m^2 T_c}}{J_{1m}}\right) \tag{22a}$$

where $J_{1m} = J_1(\alpha_m)$. Differentiate Eq. (22a) with respect to again to derive

$$(-2\nu T_{c1\nu}^{-2}-T_{c1\nu}^{-3}T_{c2\nu})\frac{e^{-\nu^2}}{\sqrt{\pi}}=\sum_{m=1}^{\infty}\left(\frac{\alpha_m^3 e^{-\alpha_m^2 T_c}}{J_{1m}}\right) \tag{23}$$

Differentiate Eq. (23) with respect to ν again to derive

$$[(2-4\nu^2)T_{c1\nu}^{-3}-6\nu T_{c1\nu}^{-4}T_{c2\nu}-3T_{c1\nu}^{-5}T_{c2\nu}^2+T_{c1\nu}^{-4}T_{c3\nu}]\frac{e^{-\nu^2}}{\sqrt{\pi}}=\sum_{m=1}^{\infty}\left(\frac{\alpha_m^5 e^{-\alpha_m^2 T_c}}{J_{1m}}\right) \tag{24}$$

Differentiate Eq. (24) with respect to ν again to derive

$$[(12\nu-8\nu^3)T_{c1\nu}^{-4}+(12-24\nu^2)T_{c1\nu}^{-5}T_{c2\nu}-30\nu\,T_{c1\nu}^{-6}T_{c2\nu}^2-15T_{c1\nu}^{-7}T_{c2\nu}^3+8\nu\,T_{c1\nu}^{-5}T_{c3\nu}+10T_{c1\nu}^{-6}T_{c2\nu}T_{c3\nu}-T_{c1\nu}^{-5}T_{c4\nu}]\frac{e^{-\nu^2}}{\sqrt{\pi}}=\sum_{m=1}^{\infty}\left(\frac{\alpha_m^7 e^{-\alpha_m^2 T_c}}{J_{1m}}\right) \tag{25}$$

Differentiate Eq. (25) with respect to ν again to derive

$$\begin{aligned}&[(-12+48\nu^2-16\nu^4)\,T_{c1\nu}^{-5}+(120\nu-80\nu^3)\,T_{c1\nu}^{-6}T_{c2\nu}\\&+(90-180\nu^2)\,T_{c1\nu}^{-7}T_{c2\nu}^2-210\nu\,T_{c1\nu}^{-8}T_{c2\nu}^3-105T_{c1\nu}^{-9}T_{c2\nu}^4\\&+(-20+40\nu^2)\,T_{c1\nu}^{-6}T_{c3\nu}+120\nu\,T_{c1\nu}^{-7}T_{c2\nu}T_{c3\nu}\\&+105T_{c1\nu}^{-8}T_{c2\nu}^2T_{c3\nu}-10T_{c1\nu}^{-7}T_{c3\nu}^2-10\nu\,T_{c1\nu}^{-6}T_{c4\nu}\\&-15T_{c1\nu}^{-7}T_{c2\nu}T_{c4\nu}+T_{c1\nu}^{-6}T_{c5\nu}]\frac{e^{-\nu^2}}{\sqrt{\pi}}=\sum_{m=1}^{\infty}\left(\frac{\alpha_m^9 e^{-\alpha_m^2 T_c}}{J_{1m}}\right)\end{aligned} \tag{26}$$

Differentiate Eq. (26) with respect to ν again to derive

$$\begin{aligned}&[(-120\nu+160\nu^3-32\nu^5)\,T_{c1\nu}^{-6}+(-180+720\nu^2-240\nu^4)T_{c1\nu}^{-7}T_{c2\nu}\\&+(1260\nu-840\nu^3)\,T_{c1\nu}^{-8}T_{c2\nu}^2+(840-1680\nu^2)T_{c1\nu}^{-9}T_{c2\nu}^3\\&-1890\nu\,T_{c1\nu}^{-10}T_{c2\nu}^4-945T_{c1\nu}^{-11}T_{c2\nu}^5+(-240\nu+160\nu^3)\,T_{c1\nu}^{-7}T_{c3\nu}\\&+(-420+840\nu^2)\,T_{c1\nu}^{-8}T_{c2\nu}T_{c3\nu}+1680\nu\,T_{c1\nu}^{-9}T_{c2\nu}^2T_{c3\nu}\\&+1260T_{c1\nu}^{-10}T_{c2\nu}^3T_{c3\nu}-140\nu\,T_{c1\nu}^{-8}T_{c3\nu}^2-280T_{c1\nu}^{-9}T_{c2\nu}T_{c3\nu}^2\\&+(30-60\nu^2)T_{c1\nu}^{-7}T_{c4\nu}-210\nu\,T_{c1\nu}^{-8}T_{c2\nu}T_{c4\nu}-210T_{c1\nu}^{-9}T_{c2\nu}^2T_{c4\nu}\\&+35T_{c1\nu}^{-8}T_{c3\nu}T_{c4\nu}+12\nu\,T_{c1\nu}^{-7}T_{c5\nu}+21T_{c1\nu}^{-8}T_{c2\nu}T_{c5\nu}-T_{c1\nu}^{-7}T_{c6\nu}]\frac{e^{-\nu^2}}{\sqrt{\pi}}\\&=\sum_{m=1}^{\infty}\left(\frac{\alpha_m^{11} e^{-\alpha_m^2 T_c}}{J_{1m}}\right)\end{aligned} \tag{27}$$

Let $\nu = \nu_h$, in Eqs. (22a) and (23) ~ (27) to obtain

$$T_{c1\nu h}=-\frac{e^{-\nu_h^2}}{\sqrt{\pi}}\left[\sum_{m=1}^{\infty}\left(\frac{\alpha_m e^{-\alpha_m^2 T_{cvh}}}{J_{1m}}\right)\right]^{-1} \tag{28}$$

$$T_{c2\nu h}=-2\nu_h T_{c1\nu h}-T_{c1\nu h}^3\left(\frac{e^{-\nu_h^2}}{\sqrt{\pi}}\right)^{-1}\sum_{m=1}^{\infty}\left(\frac{\alpha_m^3 e^{-\alpha_m^2 T_{cvh}}}{J_{1m}}\right) \tag{29}$$

$$T_{c3\nu h} = (-2 + 4\nu_h^2)T_{c1\nu h} + 6\nu_h T_{c2\nu h} + 3T_{c1\nu h}^{-1}T_{c2\nu h}^2 + T_{c1\nu h}^4\left(\frac{e^{-\nu_h^2}}{\sqrt{\pi}}\right)^{-1}\sum_{m=1}^{\infty}\left(\frac{\alpha_m^5 e^{-\alpha_m^2 T_{c\nu h}}}{J_{1m}}\right) \quad (30)$$

$$\begin{aligned} T_{c4\nu h} &= (12\nu_h - 8\nu_h^3)T_{c1\nu h} + (12 - 24\nu_h^2)T_{c2\nu h} \\ &\quad - 30\nu_h T_{c1\nu h}^{-1}T_{c2\nu h}^2 - 15T_{c1\nu h}^{-2}T_{c2\nu h}^3 + 8\nu_h T_{c3\nu h} \\ &\quad + 10T_{c1\nu h}^{-1}T_{c2\nu h}T_{c3\nu h} - T_{c1\nu h}^5\left(\frac{e^{-\nu_h^2}}{\sqrt{\pi}}\right)^{-1}\sum_{m=1}^{\infty}\left(\frac{\alpha_m^7 e^{-\alpha_m^2 T_{c\nu h}}}{J_{1m}}\right) \end{aligned} \quad (31)$$

$$\begin{aligned} T_{c5\nu h} &= (12 - 48\nu_h^2 + 16\nu_h^4)T_{c1\nu h} + (-120\nu_h + 80\nu_h^3)T_{c2\nu h} \\ &\quad + (-90 + 180\nu_h^2)T_{c1\nu h}^{-1}T_{c2\nu h}^2 + 210\nu_h T_{c1\nu h}^{-2}T_{c2\nu h}^3 \\ &\quad + 105T_{c1\nu h}^{-3}T_{c2\nu h}^4 + (20 - 40\nu_h^2)T_{c3\nu h} \\ &\quad - 120\nu_h T_{c1\nu h}^{-1}T_{c2\nu h}T_{c3\nu h} - 105T_{c1\nu h}^{-2}T_{c2\nu h}^2 T_{c3\nu h} \\ &\quad + 10T_{c1\nu h}^{-1}T_{c3\nu h}^2 + 10\nu_h T_{c4\nu h} + 15T_{c1\nu h}^{-1}T_{c2\nu h}T_{c4\nu h} \\ &\quad + T_{c1\nu h}^6\left(\frac{e^{-\nu_h^2}}{\sqrt{\pi}}\right)^{-1}\sum_{m=1}^{\infty}\left(\frac{\alpha_m^9 e^{-\alpha_m^2 T_{c\nu h}}}{J_{1m}}\right) \end{aligned} \quad (32)$$

$$\begin{aligned} T_{c6\nu h} &= (-120\nu_h + 160\nu_h^3 - 32\nu_h^5)\,T_{c1\nu h} \\ &\quad + (-180 + 720\nu_h^2 - 240\nu_h^4)\,T_{c2\nu h} \\ &\quad + (1260\nu_h - 840\nu_h^3)\,T_{c1\nu h}^{-1}T_{c2\nu h}^2 \\ &\quad + (840 - 1680\nu_h^2)\,T_{c1\nu h}^{-2}T_{c2\nu h}^3 - 1890\nu_h T_{c1\nu h}^{-3}T_{c2\nu h}^4 \\ &\quad - 945T_{c1\nu h}^{-4}T_{c2\nu h}^5 + (-240\nu_h + 160\nu_h^3)T_{c3\nu h} \\ &\quad + (-420 + 840\nu_h^2)\,T_{c1\nu h}^{-1}T_{c2\nu h}T_{c3\nu h} \\ &\quad + 1680\nu_h T_{c1\nu h}^{-2}T_{c2\nu h}^2 T_{c3\nu h} + 1260T_{c1\nu h}^{-3}T_{c2\nu h}^3 T_{c3\nu h} \\ &\quad - 140\nu_h T_{c1\nu h}^{-1}T_{c3\nu h}^2 - 280T_{c1\nu h}^{-2}T_{c2\nu h}T_{c3\nu h}^2 \\ &\quad + (30 - 60\nu_h^2)\,T_{c4\nu h} - 210\nu_h T_{c1\nu h}^{-1}T_{c2\nu h}T_{c4\nu h} \\ &\quad - 210T_{c1\nu h}^{-2}T_{c2\nu h}^2 T_{c4\nu h} + 35T_{c1\nu h}^{-1}T_{c3\nu h}T_{c4\nu h} \\ &\quad + 12\nu_h T_{c5\nu h} + 21T_{c1\nu h}^{-1}T_{c2\nu h}T_{c5\nu h} \\ &\quad - T_{c1\nu h}^7\left(\frac{e^{-\nu_h^2}}{\sqrt{\pi}}\right)^{-1}\sum_{m=1}^{\infty}\left(\frac{\alpha_m^{11} e^{-\alpha_m^2 T_{c\nu h}}}{J_{1m}}\right) \end{aligned} \quad (33)$$

The same procedure can be repeated to obtain the coefficients of terms with order higher than six in Eq. (21), while patience and carefulness are needed. Let ν_h = 1.6, which is theoretically arbitrary, Eq. (21) becomes (Fig. 3)

$$\begin{aligned} T_c &= 5.696699\times10^{-2} \\ &\quad - 4.834855\times10^{-2}(\nu - 1.6) + 2.960385\times10^{-2}(\nu - 1.6)^2 \\ &\quad - 1.583483\times10^{-2}(\nu - 1.6)^3 + 7.980194\times10^{-3}(\nu - 1.6)^4 \\ &\quad - 3.940262\times10^{-3}(\nu - 1.6)^5 + 1.973469\times10^{-3}(\nu - 1.6)^6 \\ &\quad + \cdots \end{aligned} \quad (34)$$

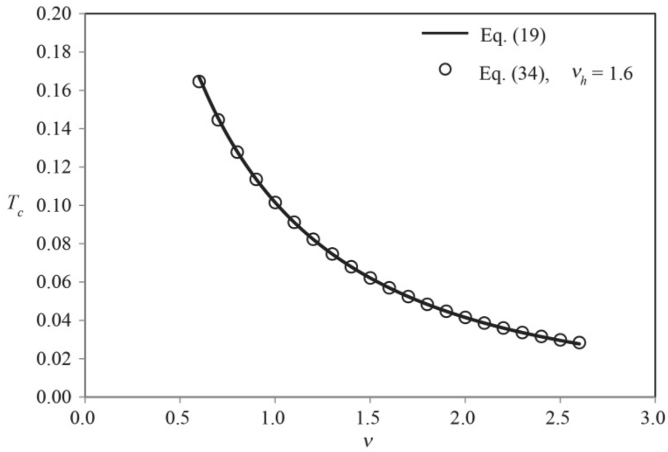

Fig. 3 T_c versus ν in a cylinder medium

T_c calculated based on the seven terms on the right hand side of Eq. (34) match perfectly with those calculated numerically from Eq. (19) for various ν.

3.2 Functional relationship between *R* and for a given *T*

If *T* in Eq. (18) is given as T_k, *R* can be calculated numerically and is sensitive to (Fig. 4). Tsai process is again used to derive the functional relationship between *R* and. Define

$$R_{Tk1\nu} = \frac{\partial R_{Tk}}{\partial \nu}, R_{Tk2\nu} = \frac{\partial^2 R_{Tk}}{\partial \nu^2}, R_{Tk3\nu} = \frac{\partial^3 R_{Tk}}{\partial \nu^3}, R_{Tk4\nu} = \frac{\partial^4 R_{Tk}}{\partial \nu^4}, R_{Tk5\nu} = \frac{\partial^5 R_{Tk}}{\partial \nu^5}, R_{Tk6\nu} = \frac{\partial^6 R_{Tk}}{\partial \nu^6}, \cdots \quad (35)$$

and express *R* at $T = T_k$ as a function of in Taylor series at the chosen ν_h.

$$\begin{aligned} R_{Tk} &= R_{Tk\nu h} + \frac{R_{Tk1\nu h}}{1!}(\nu - \nu_h) + \frac{R_{Tk2\nu h}}{2!}(\nu - \nu_h)^2 + \frac{R_{Tk3\nu h}}{3!}(\nu - \nu_h)^3 \\ &\quad + \frac{R_{Tk4\nu h}}{4!}(\nu - \nu_h)^4 + \frac{R_{Tk5\nu h}}{5!}(\nu - \nu_h)^5 + \frac{R_{Tk6\nu h}}{6!}(\nu - \nu_h)^6 \\ &\quad + \cdots \end{aligned} \quad (36)$$

where R_{Tk} is *R* at $T = T_k$. $R_{Tk\nu h}$ is R_{Tk} at $\nu = \nu_h$. $R_{Tk\nu h}$ can be calculated numerically from Eq. (18) by letting $\nu = \nu_h$ and $T = T_k$. $R_{Tk1\nu h}$, $R_{Tk2\nu h}$, $R_{Tk3\nu h}$, $R_{Tk4\nu h}$, $R_{Tk5\nu h}$ and $R_{Tk6\nu h}$, are $R_{Tk1\nu}$, $R_{Tk2\nu}$, $R_{Tk3\nu}$, $R_{Tk4\nu}$, $R_{Tk5\nu}$ and $R_{Tk6\nu}$ at $\nu = \nu_h$. Differentiate Eq. (18) with respect to ν once and let $T = T_k$ to derive

$$\frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{\nu^{2n}}{n!} = \frac{e^{-\nu^2}}{\sqrt{\pi}} = -R_{Tk1\nu}\sum_{m=1}^{\infty}\left(\frac{e^{-\alpha_m^2 T_k}}{J_{1m}}J_{1mk}\right) \quad (37)$$

or

$$-R_{Tk1\nu}^{-1}\frac{e^{-\nu^2}}{\sqrt{\pi}} = \sum_{m=1}^{\infty}\left(\frac{e^{-\alpha_m^2 T_k}}{J_{1m}}J_{1mk}\right) \quad (37a)$$

Differentiate Eq. (37a) with respect to ν again to derive

$$(2\nu\, R_{Tk1\nu}^{-2} - R_{Tk}^{-1}R_{Tk1\nu}^{-1} + R_{Tk1\nu}^{-3}R_{Tk2\nu})\frac{e^{-\nu^2}}{\sqrt{\pi}} = \sum_{m=1}^{\infty}\left(\frac{\alpha_m e^{-\alpha_m^2 T_k}}{J_{1m}} J_{0mk}\right) \quad (38)$$

where $J_{1mk} = J_1(\alpha_m R_{Tk})$, $J_{0mk} = J_0(\alpha_m R_{Tk})$, $\frac{\partial J_{1mk}}{\partial \nu} = (\alpha_m R_{Tk1\nu} J_{0mk} - R_{Tk}^{-1} R_{Tk1\nu} J_{1mk})$, and $\frac{\partial J_{0mk}}{\partial \nu} = -\alpha_m R_{Tk1\nu} J_{1mk}$. Differentiate Eq. (38) with respect to ν again to derive

$$\begin{aligned}&[(-2+4\nu^2)R_{Tk1\nu}^{-3} - 2\nu\, R_{Tk}^{-1}R_{Tk1\nu}^{-2} - R_{Tk}^{-2}R_{Tk1\nu}^{-1} + 6\nu\, R_{Tk1\nu}^{-4}R_{Tk2\nu}\\ &- R^{-1}R_{Tk1\nu}^{-3}R_{Tk2\nu} + 3R_{Tk1\nu}^{-5}R_{Tk2\nu}^{2} - R_{Tk1\nu}^{-4}R_{Tk3\nu}]\frac{e^{-\nu^2}}{\sqrt{\pi}}\\ &= \sum_{m=1}^{\infty}\left(\frac{\alpha_m^2 e^{-\alpha_m^2 T_k}}{J_{1m}} J_{1mk}\right)\end{aligned} \quad (39)$$

Differentiate Eq. (39) with respect to ν again to derive

$$\begin{aligned}&[(12\nu - 8\nu^3)R_{Tk1\nu}^{-4} + (-4+8\nu^2)R_{Tk}^{-1}R_{Tk1\nu}^{-3} + 2\nu\, R_{Tk}^{-2}R_{Tk1\nu}^{-2}\\ &+ R_{Tk}^{-3}R_{Tk1\nu}^{-1} + (12-24\nu^2)R_{Tk1\nu}^{-5}R_{Tk2\nu} + 12\nu\, R_{Tk}^{-1}R_{Tk1\nu}^{-4}R_{Tk2\nu}\\ &+ R_{Tk}^{-2}R_{Tk1\nu}^{-3}R_{Tk2\nu} - 30\nu\, R_{Tk1\nu}^{-6}R_{Tk2\nu}^{2} + 6R_{Tk}^{-1}R_{Tk1\nu}^{-5}R_{Tk2\nu}^{2}\\ &- 15R_{Tk1\nu}^{-7}R_{Tk2\nu}^{3} + 8\nu\, R_{Tk1\nu}^{-5}R_{Tk3\nu} - 2R_{Tk}^{-1}R_{Tk1\nu}^{-4}R_{Tk3\nu}\\ &+ 10R_{Tk1\nu}^{-6}R_{Tk2\nu}R_{Tk3\nu} - R_{Tk1\nu}^{-5}R_{Tk4\nu}]\frac{e^{-\nu^2}}{\sqrt{\pi}}\\ &= \sum_{m=1}^{\infty}\left(\frac{\alpha_m^3 e^{-\alpha_m^2 T_k}}{J_{1m}} J_{0mk}\right)\end{aligned} \quad (40)$$

Differentiate Eq. (40) with respect to ν again to derive

$$\begin{aligned}&[(-12+48\nu^2-16\nu^4)R_{Tk1\nu}^{-5} + (-24\nu + 16\nu^3)R_{Tk}^{-1}R_{Tk1\nu}^{-4}\\ &+ (-6+12\nu^2)R_{Tk}^{-2}R_{Tk1\nu}^{-3} + 6\nu\, R_{Tk}^{-3}R_{Tk1\nu}^{-2} + 3R_{Tk}^{-4}R_{Tk1\nu}^{-1}\\ &+ (120\nu - 80\nu^3)R_{Tk1\nu}^{-6}R_{Tk2\nu} + (-24+48\nu^2)R_{Tk}^{-1}R_{Tk1\nu}^{-5}R_{Tk2\nu}\\ &+ 18\nu\, R_{Tk}^{-2}R_{Tk1\nu}^{-4}R_{Tk2\nu} + 3R_{Tk}^{-3}R_{Tk1\nu}^{-3}R_{Tk2\nu}\\ &+ (90-180\nu^2)R_{Tk1\nu}^{-7}R_{Tk2\nu}^{2} + 60\nu\, R_{Tk}^{-1}R_{Tk1\nu}^{-6}R_{Tk2\nu}^{2}\\ &+ 9R_{Tk}^{-2}R_{Tk1\nu}^{-5}R_{Tk2\nu}^{2} - 210\nu\, R_{Tk1\nu}^{-8}R_{Tk2\nu}^{3} + 30R_{Tk}^{-1}R_{Tk1\nu}^{-7}R_{Tk2\nu}^{3}\\ &- 105R_{Tk1\nu}^{-9}R_{Tk2\nu}^{4} + (-20+40\nu^2)R_{Tk1\nu}^{-6}R_{Tk3\nu}\\ &- 16\nu\, R_{Tk}^{-1}R_{Tk1\nu}^{-5}R_{Tk3\nu} - 3R_{Tk}^{-2}R_{Tk1\nu}^{-4}R_{Tk3\nu}\\ &+ 120\nu\, R_{Tk1\nu}^{-7}R_{Tk2\nu}R_{Tk3\nu} - 20R_{Tk}^{-1}R_{Tk1\nu}^{-6}R_{Tk2\nu}R_{Tk3\nu}\\ &+ 105R_{Tk1\nu}^{-8}R_{Tk2\nu}^{2}R_{Tk3\nu} - 10R_{Tk1\nu}^{-7}R_{Tk3\nu}^{2}\\ &- 10\nu\, R_{Tk1\nu}^{-6}R_{Tk4\nu} + 2R_{Tk}^{-1}R_{Tk1\nu}^{-5}R_{Tk4\nu} - 15R_{Tk1\nu}^{-7}R_{Tk2\nu}R_{Tk4\nu}\\ &+ R_{Tk1\nu}^{-6}R_{Tk5\nu}]\frac{e^{-\nu^2}}{\sqrt{\pi}} = \sum_{m=1}^{\infty}\left(\frac{\alpha_m^4 e^{-\alpha_m^2 T_k}}{J_{1m}} J_{1mk}\right)\end{aligned} \quad (41)$$

Differentiate Eq. (41) with respect to ν again to derive

$$\begin{aligned}&[(120\nu - 160\nu^3 + 32\nu^5)R_{Tk1\nu}^{-6} + (-36+144\nu^2-48\nu^4)R_{Tk}^{-1}R_{Tk1\nu}^{-5}\\ &+ (36\nu - 24\nu^3)R_{Tk}^{-2}R_{Tk1\nu}^{-4} + (12-24\nu^2)R_{Tk}^{-3}R_{Tk1\nu}^{-3} - 18\nu\, R_{Tk}^{-4}R_{Tk1\nu}^{-2}\\ &- 9R_{Tk}^{-5}R_{Tk1\nu}^{-1} + (180-720\nu^2+240\nu^4)R_{Tk1\nu}^{-7}R_{Tk2\nu}\\ &+ (360\nu - 240\nu^3)R_{Tk}^{-1}R_{Tk1\nu}^{-6}R_{Tk2\nu} + (36-72\nu^2)R_{Tk}^{-2}R_{Tk1\nu}^{-5}R_{Tk2\nu}\\ &- 36\nu\, R_{Tk}^{-3}R_{Tk1\nu}^{-4}R_{Tk2\nu} - 9R_{Tk}^{-4}R_{Tk1\nu}^{-3}R_{Tk2\nu} + (-1260\nu + 840\nu^3)R_{Tk1\nu}^{-8}R_{Tk2\nu}^{2}\\ &+ (270-540\nu^2)R_{Tk}^{-1}R_{Tk1\nu}^{-7}R_{Tk2\nu}^{2} - 90\nu\, R_{Tk}^{-2}R_{Tk1\nu}^{-6}R_{Tk2\nu}^{2} - 18R^{-3}R_{Tk1\nu}^{-5}R_{Tk2\nu}^{2}\\ &+ (-840+1680\nu^2)R_{Tk1\nu}^{-9}R_{Tk2\nu}^{3} - 630\nu\, R_{Tk}^{-1}R_{Tk1\nu}^{-8}R_{Tk2\nu}^{3} - 45R_{Tk}^{-2}R_{Tk1\nu}^{-7}R_{Tk2\nu}^{3}\\ &+ 1890\nu\, R_{Tk1\nu}^{-10}R_{Tk2\nu}^{4} - 315R_{Tk}^{-1}R_{Tk1\nu}^{-9}R_{Tk2\nu}^{4} + 945R_{Tk1\nu}^{-11}R_{Tk2\nu}^{5}\\ &+ (240\nu - 160\nu^3)R_{Tk1\nu}^{-7}R_{Tk3\nu} + (-60+120\nu^2)R_{Tk}^{-1}R_{Tk1\nu}^{-6}R_{Tk3\nu}\\ &+ 24\nu\, R_{Tk}^{-2}R_{Tk1\nu}^{-5}R_{Tk3\nu} + 6R_{Tk}^{-3}R_{Tk1\nu}^{-4}R_{Tk3\nu} + (420-840\nu^2)R_{Tk1\nu}^{-8}R_{Tk2\nu}R_{Tk3\nu}\\ &+ 360\nu\, R_{Tk}^{-1}R_{Tk1\nu}^{-7}R_{Tk2\nu}R_{Tk3\nu} + 30R_{Tk}^{-2}R_{Tk1\nu}^{-6}R_{Tk2\nu}R_{Tk3\nu}\\ &- 1680\nu\, R_{Tk1\nu}^{-9}R_{Tk2\nu}^{2}R_{Tk3\nu} + 315R_{Tk}^{-1}R_{Tk1\nu}^{-8}R_{Tk2\nu}^{2}R_{Tk3\nu} - 1260R_{Tk1\nu}^{-10}R_{Tk2\nu}^{3}R_{Tk3\nu}\\ &+ 140\nu\, R_{Tk1\nu}^{-8}R_{Tk3\nu}^{2} - 30R_{Tk}^{-1}R_{Tk1\nu}^{-7}R_{Tk3\nu}^{2} + 280R_{Tk1\nu}^{-9}R_{Tk2\nu}R_{Tk3\nu}^{2}\\ &+ (-30+60\nu^2)R_{Tk1\nu}^{-7}R_{Tk4\nu} - 30\nu\, R_{Tk}^{-1}R_{Tk1\nu}^{-6}R_{Tk4\nu} - 3R_{Tk}^{-2}R_{Tk1\nu}^{-5}R_{Tk4\nu}\\ &+ 210\nu\, R_{Tk1\nu}^{-8}R_{Tk2\nu}R_{Tk4\nu} - 45R_{Tk}^{-1}R_{Tk1\nu}^{-7}R_{Tk2\nu}R_{Tk4\nu} + 210R_{Tk1\nu}^{-9}R_{Tk2\nu}^{2}R_{Tk4\nu}\\ &- 35R_{Tk1\nu}^{-8}R_{Tk3\nu}R_{Tk4\nu} - 12\nu\, R_{Tk1\nu}^{-7}R_{Tk5\nu} + 3R_{Tk}^{-1}R_{Tk1\nu}^{-6}R_{Tk5\nu}\\ &- 21R_{Tk1\nu}^{-8}R_{Tk2\nu}R_{Tk5\nu} + R_{Tk1\nu}^{-7}R_{Tk6\nu}]\frac{e^{-\nu^2}}{\sqrt{\pi}} = \sum_{m=1}^{\infty}\left(\frac{\alpha_m^5 e^{-\alpha_m^2 T_k}}{J_{1m}} J_{0mk}\right)\end{aligned} \quad (42)$$

Let $\nu = \nu_h$ in Eqs. (37a, 38, 39, 40, 41, 42) to obtain $R_{Tk1\nu h}$, $R_{Tk2\nu h}$, $R_{Tk3\nu h}$, $R_{Tk4\nu h}$, $R_{Tk5\nu h}$ and $R_{Tk6\nu h}$.

$$R_{Tk1\nu h} = -\frac{e^{-\nu_h^2}}{\sqrt{\pi}}\left[\sum_{m=1}^{\infty}\left(\frac{e^{-\alpha_m^2 T_k}}{J_{1m}} J_{1mkh}\right)\right]^{-1} \quad (43)$$

$$\begin{aligned}R_{Tk2\nu h} &= (-2\nu_h R_{Tk1\nu h} + R_{Tk\nu h}^{-1}R_{Tk1\nu h}^{2})\\ &+ R_{Tk1\nu h}^{3}\left(\frac{e^{-\nu_h^2}}{\sqrt{\pi}}\right)^{-1}\sum_{m=1}^{\infty}\left(\frac{\alpha_m e^{-\alpha_m^2 T_k}}{J_{1m}} J_{0mkh}\right)\end{aligned} \quad (44)$$

where $J_{0mkh} = J_0(\alpha_m R_{Tk\nu h})$, and $J_{1mkh} = J_1(\alpha_m R_{Tk\nu h})$.

$$\begin{aligned}R_{Tk3\nu h} &= (-2+4\nu_h^2)R_{Tk1\nu h} - 2\nu_h R_{Tk\nu h}^{-1}R_{Tk1\nu h}^{2} - R_{Tk\nu h}^{-2}R_{Tk1\nu h}^{3}\\ &+ 6\nu_h R_{Tk2\nu h} - R_{Tk\nu h}^{-1}R_{Tk1\nu h}R_{Tk2\nu h} + 3R_{Tk1\nu h}^{-1}R_{Tk2\nu h}^{2}\\ &- R_{Tk1\nu h}^{4}\left(\frac{e^{-\nu_h^2}}{\sqrt{\pi}}\right)^{-1}\sum_{m=1}^{\infty}\left(\frac{\alpha_m^2 e^{-\alpha_m^2 T_k}}{J_{1m}} J_{1mkh}\right)\end{aligned} \quad (45)$$

$$\begin{aligned}R_{Tk4\nu h} &= (12\nu_h - 8\nu_h^3)R_{Tk1\nu h} + (-4+8\nu_h^2)R_{Tk\nu h}^{-1}R_{Tk1\nu h}^{2}\\ &+ 2\nu_h R_{Tk\nu h}^{-2}R_{Tk1\nu h}^{3} + R_{Tk\nu h}^{-3}R_{Tk1\nu h}^{4} + (12-24\nu_h^2)R_{Tk2\nu h}\\ &+ 12\nu_h R_{Tk\nu h}^{-1}R_{Tk1\nu h}R_{Tk2\nu h} + R_{Tk\nu h}^{-2}R_{Tk1\nu h}^{2}R_{Tk2\nu h}\\ &- 30\nu_h R_{Tk1\nu h}^{-1}R_{Tk2\nu h}^{2} + 6R_{Tk\nu h}^{-1}R_{Tk2\nu h}^{2} - 15R_{Tk1\nu h}^{-2}R_{Tk2\nu h}^{3}\\ &+ 8\nu_h R_{Tk3\nu h} - 2R_{Tk\nu h}^{-1}R_{Tk1\nu h}R_{Tk3\nu h} + 10R_{Tk1\nu h}^{-1}R_{Tk2\nu h}R_{Tk3\nu h}]\\ &- R_{Tk1\nu h}^{5}\left(\frac{e^{-\nu_h^2}}{\sqrt{\pi}}\right)^{-1}\sum_{m=1}^{\infty}\left(\frac{\alpha_m^3 e^{-\alpha_m^2 T_k}}{J_{1m}} J_{0mkh}\right)\end{aligned} \quad (46)$$

$$\begin{aligned}R_{Tk5\nu h} &= (12 - 48\nu_h^2 + 16\nu_h^4)R_{Tk1\nu h} + (24\nu_h - 16\nu_h^3)R_{Tk\nu h}^{-1}R_{Tk1\nu h}^2 \\ &+ (6 - 12\nu_h^2)R_{Tk\nu h}^{-2}R_{Tk1\nu h}^3 - 6\nu_h R_{Tk\nu h}^{-3}R_{Tk1\nu h}^4 - 3R_{Tk\nu h}^{-4}R_{Tk1\nu h}^5 \\ &+ (-120\nu_h + 80\nu_h^3)R_{Tk2\nu h} + (24 - 48\nu_h^2)R_{Tk\nu h}^{-1}R_{Tk1\nu h}R_{Tk2\nu h} \\ &- 18\nu_h R_{Tk\nu h}^{-2}R_{Tk1\nu h}^2R_{Tk2\nu h} - 3R_{Tk\nu h}^{-3}R_{Tk1\nu h}^3R_{Tk2\nu h} \\ &+ (-90 + 180\nu_h^2)R_{Tk1\nu h}^{-1}R_{Tk2\nu h}^2 - 60\nu_h R_{Tk\nu h}^{-1}R_{Tk2\nu h}^2 \\ &- 9R_{Tk\nu h}^{-2}R_{Tk1\nu h}R_{Tk2\nu h}^2 + 210\nu_h R_{Tk1\nu h}^{-2}R_{Tk2\nu h}^3 \\ &- 30R_{Tk\nu h}^{-1}R_{Tk1\nu h}^{-1}R_{Tk2\nu h}^3 + 105R_{Tk1\nu h}^{-3}R_{Tk2\nu h}^4 \\ &+ (20 - 40\nu_h^2)R_{Tk3\nu h} + 16\nu_h R_{Tk\nu h}^{-1}R_{Tk1\nu h}R_{Tk3\nu h} \\ &+ 3R_{Tk\nu h}^{-2}R_{Tk1\nu h}^2R_{Tk3\nu h} - 120\nu_h R_{Tk1\nu h}^{-1}R_{Tk2\nu h}R_{Tk3\nu h} \\ &+ 20R_{Tk\nu h}^{-1}R_{Tk2\nu h}R_{Tk3\nu h} - 105R_{Tk1\nu h}^{-2}R_{Tk2\nu h}^2R_{Tk3\nu h} \\ &+ 10R_{Tk1\nu h}^{-1}R_{Tk3\nu h}^2 + 10\nu_h R_{Tk4\nu h} - 2R_{Tk\nu h}^{-1}R_{Tk1\nu h}R_{Tk4\nu h} \\ &+ 15R_{Tk1\nu h}^{-1}R_{Tk2\nu h}R_{Tk4\nu h} + R_{Tk1\nu h}^6\left(\frac{e^{-\nu_h^2}}{\sqrt{\pi}}\right)^{-1}\sum_{m=1}^{\infty}\left(\frac{\alpha_m^4 e^{-\alpha_m^2 T_k}}{J_{1m}}J_{1mkh}\right)\end{aligned} \tag{47}$$

$$\begin{aligned}R_{Tk6\nu h} &= (-120\nu_h + 160\nu_h^3 - 32\nu_h^5)R_{Tk1\nu h} \\ &+ (36 - 144\nu_h^2 + 48\nu_h^4)R_{Tk\nu h}^{-1}R_{Tk1\nu h}^2 + (-36\nu_h + 24\nu_h^3)R_{Tk\nu h}^{-2}R_{Tk1\nu h}^3 \\ &+ (-12 + 24\nu_h^2)R_{Tk\nu h}^{-3}R_{Tk1\nu h}^4 + 18\nu_h R_{Tk\nu h}^{-4}R_{Tk1\nu h}^5 \\ &+ 9R_{Tk\nu h}^{-5}R_{Tk1\nu h}^6 + (-180 + 720\nu_h^2 - 240\nu_h^4)R_{Tk2\nu h} \\ &+ (-360\nu_h + 240\nu_h^3)R_{Tk\nu h}^{-1}R_{Tk1\nu h}R_{Tk2\nu h} \\ &+ (-36 + 72\nu_h^2)R_{Tk\nu h}^{-2}R_{Tk1\nu h}^2R_{Tk2\nu h} + 36\nu_h R_{Tk\nu h}^{-3}R_{Tk1\nu h}^3R_{Tk2\nu h} \\ &+ 9R_{Tk\nu h}^{-4}R_{Tk1\nu h}^4R_{Tk2\nu h} + (1260\nu_h - 840\nu_h^3)R_{Tk1\nu h}^{-1}R_{Tk2\nu h}^2 \\ &+ (-270 + 540\nu_h^2)R_{Tk\nu h}^{-1}R_{Tk2\nu h}^2 + 90\nu_h R_{Tk\nu h}^{-2}R_{Tk1\nu h}R_{Tk2\nu h}^2 \\ &+ 18R_{Tk\nu h}^{-3}R_{Tk1\nu h}^2R_{Tk2\nu h}^2 + (840 - 1680\nu_h^2)R_{Tk1\nu h}^{-2}R_{Tk2\nu h}^3 \\ &+ 630\nu_h R_{Tk\nu h}^{-1}R_{Tk1\nu h}^{-1}R_{Tk2\nu h}^3 + 45R_{Tk\nu h}^{-2}R_{Tk2\nu h}^3 - 1890\nu_h R_{Tk1\nu h}^{-3}R_{Tk2\nu h}^4 \\ &+ 315R_{Tk\nu h}^{-1}R_{Tk1\nu h}^{-2}R_{Tk2\nu h}^4 - 945R_{Tk1\nu h}^{-4}R_{Tk2\nu h}^5 \\ &+ (-240\nu_h + 160\nu_h^3)R_{Tk3\nu h} + (60 - 120\nu_h^2)R_{Tk\nu h}^{-1}R_{Tk1\nu h}R_{Tk3\nu h} \\ &- 24\nu_h R_{Tk\nu h}^{-2}R_{Tk1\nu h}^2R_{Tk3\nu h} - 6R_{Tk\nu h}^{-3}R_{Tk1\nu h}^3R_{Tk3\nu h} \\ &+ (-420 + 840\nu_h^2)R_{Tk1\nu h}^{-1}R_{Tk2\nu h}R_{Tk3\nu h} - 360\nu_h R_{Tk\nu h}^{-1}R_{Tk2\nu h}R_{Tk3\nu h} \\ &- 30R_{Tk\nu h}^{-2}R_{Tk1\nu h}R_{Tk2\nu h}R_{Tk3\nu h} + 1680\nu_h R_{Tk1\nu h}^{-2}R_{Tk2\nu h}^2R_{Tk3\nu h} \\ &- 315R_{Tk\nu h}^{-1}R_{Tk1\nu h}^{-1}R_{Tk2\nu h}^2R_{Tk3\nu h} + 1260R_{Tk1\nu h}^{-3}R_{Tk2\nu h}^3R_{Tk3\nu h} \\ &- 140\nu_h R_{Tk1\nu h}^{-1}R_{Tk3\nu h}^2 + 30R_{Tk\nu h}^{-1}R_{Tk3\nu h}^2 - 280R_{Tk1\nu h}^{-2}R_{Tk2\nu h}R_{Tk3\nu h}^2 \\ &+ (30 - 60\nu_h^2)R_{Tk4\nu h} + 30\nu_h R_{Tk\nu h}^{-1}R_{Tk1\nu h}R_{Tk4\nu h} + 3R_{Tk\nu h}^{-2}R_{Tk1\nu h}^2R_{Tk4\nu h} \\ &- 210\nu_h R_{Tk1\nu h}^{-1}R_{Tk2\nu h}R_{Tk4\nu h} + 45R_{Tk\nu h}^{-1}R_{Tk2\nu h}R_{Tk4\nu h} \\ &- 210R_{Tk1\nu h}^{-2}R_{Tk2\nu h}^2R_{Tk4\nu h} + 35R_{Tk1\nu h}^{-1}R_{Tk3\nu h}R_{Tk4\nu h} \\ &+ 12\nu_h R_{Tk5\nu h} - 3R_{Tk\nu h}^{-1}R_{Tk1\nu h}R_{Tk5\nu h} + 21R_{Tk1\nu h}^{-1}R_{Tk2\nu h}R_{Tk5\nu h} \\ &+ R_{Tk1\nu h}^7\left(\frac{e^{-\nu_h^2}}{\sqrt{\pi}}\right)^{-1}\sum_{m=1}^{\infty}\left(\frac{\alpha_m^5 e^{-\alpha_m^2 T_k}}{J_{1m}}J_{0mkh}\right)\end{aligned} \tag{48}$$

The same procedure can be repeated to obtain the coefficients of terms with order higher than six in Eq. (36), while a lot of patience and carefulness are needed.

ν_h and T_k in Eq. (36) are theoretically arbitrary, only that T_k must be smaller than the T_c corresponding to ν_h. Let $\nu_h = 1.6$ again, the corresponding T_c is 5.696699×10^{-2}, and let $T_k = 3 \times 10^{-2}$. The corresponding $R_{Tk\nu h}$ calculated numerically from Eq. (18) is 4.027017×10^{-1}. Eq. (36) becomes (Fig. 4)

$$\begin{aligned}R_{Tk} &= 4.027017\times10^{-1} \\ &-3.688787\times10^{-1}(\nu-1.6) - 9.874079\times10^{-3}(\nu-1.6)^2 \\ &-1.033423\times10^{-2}(\nu-1.6)^3 - 7.331197\times10^{-3}(\nu-1.6)^4 \\ &-7.566221\times10^{-3}(\nu-1.6)^5 - 7.096703\times10^{-3}(\nu-1.6)^6 + \cdots\end{aligned} \tag{49}$$

R_{Tk} calculated based on the seven terms on the right hand side of Eq. (49) match excellently with those calculated numerically from Eq. (18).

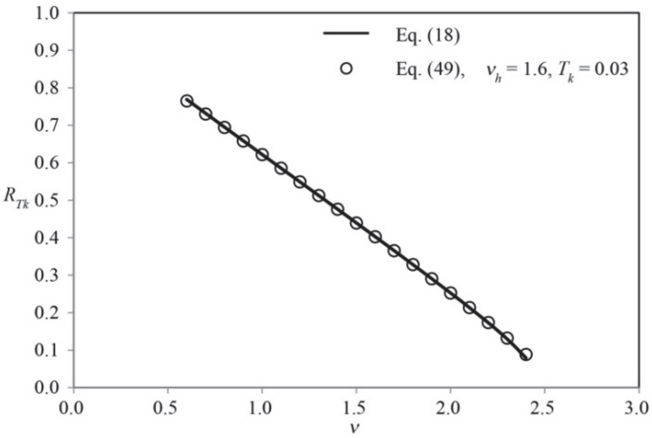

Fig. 4 R_{Tk} versus ν in a cylinder medium

3.3 Functional relationship between R and T for arbitrary ν

R depends on ν and T and can be calculated numerically from Eq. (18) (Fig. 5). Tsai process is again used to derive the functional relationship between R and T for arbitrary. Define

$$R_{1T} = \frac{\partial R}{\partial T}, R_{2T} = \frac{\partial^2 R}{\partial T^2}, R_{3T} = \frac{\partial^3 R}{\partial T^3}, R_{4T} = \frac{\partial^4 R}{\partial T^4}, \\ R_{5T} = \frac{\partial^5 R}{\partial T^5}, R_{6T} = \frac{\partial^6 R}{\partial T^6}, \cdots \tag{50}$$

and express R as a function of T in Taylor series at T_k.

$$\begin{aligned}R &= R_{Tk} + \frac{R_{1Tk}}{1!}(T - T_k) + \frac{R_{2Tk}}{2!}(T - T_k)^2 + \frac{R_{3Tk}}{3!}(T - T_k)^3 \\ &+ \frac{R_{4Tk}}{4!}(T - T_k)^4 + \frac{R_{5Tk}}{5!}(T - T_k)^5 + \frac{R_{6Tk}}{6!}(T - T_k)^6 + \cdots\end{aligned} \tag{51}$$

where R_{1Tk}, R_{2Tk}, R_{3Tk}, R_{4Tk}, R_{5Tk} and R_{6Tk} are R_{1T}, R_{2T}, R_{3T}, R_{4T}, R_{5T} and R_{6T}, at $T = T_k$. Define

$$S_I = \begin{cases} \sum_{m=1}^{\infty} \left(\frac{\alpha_m^I e^{-\alpha_m^2 T}}{J_{1m}} J_{1mR} \right), \ \frac{\partial S_I}{\partial T} = -S_{I+2} + R_{1T} S_{I+1} - R^{-1} R_{1T} S_I, \ I: \text{even integer} \\ \sum_{m=1}^{\infty} \left(\frac{\alpha_m^I e^{-\alpha_m^2 T}}{J_{1m}} J_{0mR} \right), \ \frac{\partial S_I}{\partial T} = -S_{I+2} - R_{1T} S_{I+1}, \ I: \text{odd integer} \end{cases} \tag{52}$$

Differentiate Eq. (18) with respect to T once to derive

$$R_{1T} S_0 = -S_1 \tag{53}$$

Differentiate Eq. (53) with respect to T again to derive

$$R_{2T} S_0 = R^{-1} R_{1T}^2 S_0 - R_{1T}^2 S_1 + 2R_{1T} S_2 + S_3 \tag{54}$$

Differentiate Eq. (54) with respect to T again to derive

$$R_{3T} S_0 = (-2R^{-2} R_{1T}^3 + 3R^{-1} R_{1T} R_{2T})\, S_0 + (R^{-1} R_{1T}^3 - 3R_{1T} R_{2T})\, S_1 + (-3R^{-1} R_{1T}^2 + R_{1T}^3 + 3R_{2T})\, S_2 + 3R_{1T}^2 S_3 - 3R_{1T} S_4 - S_5 \tag{55}$$

Differentiate Eq. (55) with respect to T again to derive

$$R_{4T} S_0 = (6R^{-3} R_{1T}^4 - 12R^{-2} R_{1T}^2 R_{2T} + 3R^{-1} R_{2T}^2 + 4R^{-1} R_{1T} R_{3T}) S_0 + (-3R^{-2} R_{1T}^4 + 6R^{-1} R_{1T}^2 R_{2T} - 3R_{2T}^2 - 4R_{1T} R_{3T}) S_1 + (8R^{-2} R_{1T}^3 - 2R^{-1} R_{1T}^4 - 12R^{-1} R_{1T} R_{2T} + 6R_{1T}^2 R_{2T} + 4R_{3T}) S_2 + (-4R^{-1} R_{1T}^3 + R_{1T}^4 + 12R_{1T} R_{2T}) S_3 + (6R^{-1} R_{1T}^2 - 4R_{1T}^3 - 6R_{2T}) S_4 - 6R_{1T}^2 S_5 + 4R_{1T} S_6 + S_7 \tag{56}$$

Differentiate Eq. (56) with respect to T again to derive

$$R_{5T} S_0 = (-24R^{-4} R_{1T}^5 + 60R^{-3} R_{1T}^3 R_{2T} - 30R^{-2} R_{1T} R_{2T}^2 - 20R^{-2} R_{1T}^2 R_{3T} + 10R^{-1} R_{2T} R_{3T} + 5R^{-1} R_{1T} R_{4T}) S_0 + (12R^{-3} R_{1T}^5 - 30R^{-2} R_{1T}^3 R_{2T} + 15R^{-1} R_{1T} R_{2T}^2 + 10R^{-1} R_{1T}^2 R_{3T} - 10R_{2T} R_{3T} - 5R_{1T} R_{4T})\, S_1 + (-30R^{-3} R_{1T}^4 + 7R^{-2} R_{1T}^5 + 60R^{-2} R_{1T}^2 R_{2T} - 20R^{-1} R_{1T}^3 R_{2T} - 15R^{-1} R_{2T}^2 + 15R_{1T} R_{2T}^2 - 20R^{-1} R_{1T} R_{3T} + 10R_{1T}^2 R_{3T} + 5R_{4T})\, S_2 + (15R^{-2} R_{1T}^4 - 2R^{-1} R_{1T}^5 - 30R^{-1} R_{1T}^2 R_{2T} + 10R_{1T}^3 R_{2T} + 15R_{2T}^2 + 20R_{1T} R_{3T}) S_3 + (-20R^{-2} R_{1T}^3 + 10R^{-1} R_{1T}^4 - R_{1T}^5 + 30R^{-1} R_{1T} R_{2T} - 30R_{1T}^2 R_{2T} - 10R_{3T}) S_4 + (+10R^{-1} R_{1T}^3 - 5R_{1T}^4 - 30R_{1T} R_{2T})\, S_5 + (-10R^{-1} R_{1T}^2 + 10R_{1T}^3 + 10R_{2T})\, S_6 + 10R_{1T}^2 S_7 - 5R_{1T} S_8 - S_9 \tag{57}$$

Differentiate Eq.(57) with respect to T again to derive

$$R_{6T} S_0 = (120R^{-5} R_{1T}^6 - 360R^{-4} R_{1T}^4 R_{2T} + 270R^{-3} R_{1T}^2 R_{2T}^2 - 30R^{-2} R_{2T}^3 + 120R^{-3} R_{1T}^3 R_{3T} - 120R^{-2} R_{1T} R_{2T} R_{3T} + 10R^{-1} R_{3T}^2 - 30R^{-2} R_{1T}^2 R_{4T} + 15R^{-1} R_{2T} R_{4T} + 6R^{-1} R_{1T} R_{5T}) S_0 + (-60R^{-4} R_{1T}^6 + 180R^{-3} R_{1T}^4 R_{2T} - 135R^{-2} R_{1T}^2 R_{2T}^2 + 15R^{-1} R_{2T}^3 - 60R^{-2} R_{1T}^3 R_{3T} + 60R^{-1} R_{1T} R_{2T} R_{3T} - 10R_{3T}^2 - 15R_{2T} R_{4T} + 15R^{-1} R_{1T}^2 R_{4T} - 6R_{1T} R_{5T}) S_1 + (144R^{-4} R_{1T}^5 - 33R^{-3} R_{1T}^6 - 360R^{-3} R_{1T}^3 R_{2T} + 105R^{-2} R_{1T}^4 R_{2T} + 180R^{-2} R_{1T} R_{2T}^2 - 90R^{-1} R_{1T}^2 R_{2T}^2 + 15R_{2T}^3 + 120R^{-2} R_{1T}^2 R_{3T} - 40R^{-1} R_{1T}^3 R_{3T} - 60R^{-1} R_{2T} R_{3T} + 60R_{1T} R_{2T} R_{3T} - 30R^{-1} R_{1T} R_{4T} + 15R_{1T}^2 R_{4T} + 6R_{5T})\, S_2 + (-72R^{-3} R_{1T}^5 + 9R^{-2} R_{1T}^6 + 180R^{-2} R_{1T}^3 R_{2T} - 30R^{-1} R_{1T}^4 R_{2T} - 90R^{-1} R_{1T} R_{2T}^2 + 45R_{1T}^2 R_{2T}^2 - 60R^{-1} R_{1T}^2 R_{3T} + 20R_{1T}^3 R_{3T} + 60R_{2T} R_{3T} + 30R_{1T} R_{4T}) S_3 + (90R^{-3} R_{1T}^4 - 42R^{-2} R_{1T}^5 + 3R^{-1} R_{1T}^6 - 180R^{-2} R_{1T}^2 R_{2T} + 120R^{-1} R_{1T}^3 R_{2T} - 15R_{1T}^4 R_{2T} + 45R^{-1} R_{2T}^2 - 90R_{1T} R_{2T}^2 + 60R^{-1} R_{1T} R_{3T} - 60R_{1T}^2 R_{3T} - 15R_{4T}) S_4 + (-45R^{-2} R_{1T}^4 + 12R^{-1} R_{1T}^5 - R_{1T}^6 + 90R^{-1} R_{1T}^2 R_{2T} - 60R_{1T}^3 R_{2T} - 45R_{2T}^2 - 60R_{1T} R_{3T}) S_5 + (40R^{-2} R_{1T}^3 - 30R^{-1} R_{1T}^4 + 6R_{1T}^5 - 60R^{-1} R_{1T} R_{2T} + 90R_{1T}^2 R_{2T} + 20R_{3T})\, S_6 + (-20R^{-1} R_{1T}^3 + 15R_{1T}^4 + 60R_{1T} R_{2T}) S_7 + (15R^{-1} R_{1T}^2 - 20R_{1T}^3 - 15R_{2T}) S_8 - 15R_{1T}^2 S_9 + 6R_{1T} S_{10} + S_{11} \tag{58}$$

For those coefficients of Eq. (51), let ν be an arbitrary constant and $T = T_k$ in Eq. (36) to calculate R_{Tk} and in Eqs. (53) ~ (58) to obtain R_{1Tk}, R_{2Tk}, R_{3Tk}, R_{4Tk}, R_{5Tk} and R_{6Tk}.

$$R_{1Tk} = -S_{1k} / S_{0k} \tag{59}$$

$$R_{2Tk} = (R_{Tk}^{-1} R_{1Tk}^2 S_{0k} - R_{1Tk}^2 S_{1k} + 2R_{1Tk} S_{2k} + S_{3k}) / S_{0k} \tag{60}$$

$$R_{3Tk} = [(-2R_{Tk}^{-2} R_{1Tk}^3 + 3R_{Tk}^{-1} R_{1Tk} R_{2Tk}) S_{0k} + (R_{Tk}^{-1} R_{1Tk}^3 - 3R_{1Tk} R_{2Tk}) S_{1k} + (-3R_{Tk}^{-1} R_{1Tk}^2 + R_{1Tk}^3 + 3R_{2Tk}) S_{2k} + 3R_{1Tk}^2 S_{3k} - 3R_{1Tk} S_{4k} - S_{5k}] / S_{0k} \tag{61}$$

$$
\begin{aligned}
R_{4Tk} = {} & [(6R_{Tk}^{-3}R_{1Tk}^{4} - 12R_{Tk}^{-2}R_{1Tk}^{2}R_{2Tk} + 3R_{Tk}^{-1}R_{2Tk}^{2} + 4R_{Tk}^{-1}R_{1Tk}R_{3Tk})S_{0k} \\
& + (-3R_{Tk}^{-2}R_{1Tk}^{4} + 6R_{Tk}^{-1}R_{1Tk}^{2}R_{2Tk} - 3R_{2Tk}^{2} - 4R_{1Tk}R_{3Tk})S_{1k} \\
& + (8R_{Tk}^{-2}R_{1Tk}^{3} - 2R_{Tk}^{-1}R_{1Tk}^{4} - 12R_{Tk}^{-1}R_{1Tk}R_{2Tk} + 6R_{1Tk}^{2}R_{2Tk} + 4R_{3Tk})S_{2k} \\
& + (-4R_{Tk}^{-1}R_{1Tk}^{3} + R_{1Tk}^{4} + 12R_{1Tk}R_{2Tk})S_{3k} + (6R_{Tk}^{-1}R_{1Tk}^{2} - 4R_{1Tk}^{3} \\
& - 6R_{2Tk})S_{4k} - 6R_{1Tk}^{2}S_{5k} + 4R_{1Tk}S_{6k} + S_{7k}] / S_{0k}
\end{aligned} \tag{62}
$$

$$
\begin{aligned}
R_{5Tk} = {} & [(-24R_{Tk}^{-4}R_{1Tk}^{5} + 60R_{Tk}^{-3}R_{1Tk}^{3}R_{2Tk} - 30R_{Tk}^{-2}R_{1Tk}R_{2Tk}^{2} \\
& - 20R_{Tk}^{-2}R_{1Tk}^{2}R_{3Tk} + 10R_{Tk}^{-1}R_{2Tk}R_{3Tk} + 5R_{Tk}^{-1}R_{1Tk}R_{4Tk})S_{0} \\
& + (12R_{Tk}^{-3}R_{1Tk}^{5} - 30R_{Tk}^{-2}R_{1Tk}^{3}R_{2Tk} + 15R_{Tk}^{-1}R_{1Tk}R_{2Tk}^{2} \\
& + 10R_{Tk}^{-1}R_{1Tk}^{2}R_{3Tk} - 10R_{2Tk}R_{3Tk} - 5R_{1Tk}R_{4Tk})S_{1k} \\
& + (-30R_{Tk}^{-3}R_{1Tk}^{4} + 7R_{Tk}^{-2}R_{1Tk}^{5} + 60R_{Tk}^{-2}R_{1Tk}^{2}R_{2Tk} \\
& - 20R_{Tk}^{-1}R_{1Tk}^{3}R_{2Tk} - 15R_{Tk}^{-1}R_{2Tk}^{2} + 15R_{1Tk}R_{2Tk}^{2} \\
& - 20R_{Tk}^{-1}R_{1Tk}R_{3Tk} + 10R_{1Tk}^{2}R_{3Tk} + 5R_{4Tk})S_{2k} \\
& + (15R_{Tk}^{-2}R_{1Tk}^{4} - 2R_{Tk}^{-1}R_{1Tk}^{5} - 30R_{Tk}^{-1}R_{1Tk}^{2}R_{2Tk} \\
& + 10R_{1Tk}^{3}R_{2Tk} + 15R_{2Tk}^{2} + 20R_{1Tk}R_{3Tk})S_{3k} \\
& + (-20R_{Tk}^{-2}R_{1Tk}^{3} + 10R_{Tk}^{-1}R_{1Tk}^{4} - R_{1Tk}^{5} + 30R_{Tk}^{-1}R_{1Tk}R_{2Tk} \\
& - 30R_{1Tk}^{2}R_{2Tk} - 10R_{3Tk})S_{4k} + (10R_{Tk}^{-1}R_{1Tk}^{3} - 5R_{1Tk}^{4} \\
& - 30R_{1Tk}R_{2Tk})S_{5k} + (-10R_{Tk}^{-1}R_{1Tk}^{2} + 10R_{1Tk}^{3}10R_{2Tk})S_{6k} \\
& + 10R_{1Tk}^{2}S_{7k} - 5R_{1Tk}S_{8k} - S_{9k}] / S_{0k}
\end{aligned} \tag{63}
$$

$$
\begin{aligned}
R_{6Tk} = {} & [(120R_{Tk}^{-5}R_{1Tk}^{6} - 360R_{Tk}^{-4}R_{1Tk}^{4}R_{2Tk} + 270R_{Tk}^{-3}R_{1Tk}^{2}R_{2Tk}^{2} \\
& - 30R_{Tk}^{-2}R_{2Tk}^{3} + 120R_{Tk}^{-3}R_{1Tk}^{3}R_{3Tk} - 120R_{Tk}^{-2}R_{1Tk}R_{2Tk}R_{3Tk} + 10R_{Tk}^{-1}R_{3Tk}^{2} \\
& - 30R_{Tk}^{-2}R_{1Tk}^{2}R_{4Tk} + 15R_{Tk}^{-1}R_{2Tk}R_{4Tk} + 6R_{Tk}^{-1}R_{1Tk}R_{5Tk})S_{0} \\
& + (-60R_{Tk}^{-4}R_{1Tk}^{6} + 180R_{Tk}^{-3}R_{1Tk}^{4}R_{2Tk} - 135R_{Tk}^{-2}R_{1Tk}^{2}R_{2Tk}^{2} + 15R_{Tk}^{-1}R_{2Tk}^{3} \\
& - 60R_{Tk}^{-2}R_{1Tk}^{3}R_{3Tk} + 60R_{Tk}^{-1}R_{1Tk}R_{2Tk}R_{3Tk} - 10R_{3Tk}^{2} - 15R_{2Tk}R_{4Tk} \\
& + 15R_{Tk}^{-1}R_{1Tk}^{2}R_{4Tk} - 6R_{1Tk}R_{5Tk})S_{1k} + (144R_{Tk}^{-4}R_{1Tk}^{5} - 33R_{Tk}^{-3}R_{1Tk}^{6} \\
& - 360R_{Tk}^{-3}R_{1Tk}^{3}R_{2Tk} + 105R_{Tk}^{-2}R_{1Tk}^{4}R_{2Tk} + 180R_{Tk}^{-2}R_{1Tk}R_{2Tk}^{2} \\
& - 90R_{Tk}^{-1}R_{1Tk}^{2}R_{2Tk}^{2} + 15R_{2Tk}^{3} + 120R_{Tk}^{-2}R_{1Tk}^{2}R_{3Tk} - 40R_{Tk}^{-1}R_{1Tk}^{3}R_{3Tk} \\
& - 60R_{Tk}^{-1}R_{2Tk}R_{3Tk} + 60R_{1Tk}R_{2Tk}R_{3Tk} - 30R_{Tk}^{-1}R_{1Tk}R_{4Tk} + 15R_{1Tk}^{2}R_{4Tk} \\
& + 6R_{5Tk})S_{2k} + (-72R_{Tk}^{-3}R_{1Tk}^{5} + 9R_{Tk}^{-2}R_{1Tk}^{6} + 180R_{Tk}^{-2}R_{1Tk}^{3}R_{2Tk} \\
& - 30R_{Tk}^{-1}R_{1Tk}^{4}R_{2Tk} - 90R_{Tk}^{-1}R_{1Tk}R_{2Tk}^{2} + 45R_{1Tk}^{2}R_{2Tk}^{2} - 60R_{Tk}^{-1}R_{1Tk}^{2}R_{3Tk} \\
& + 20R_{1Tk}^{3}R_{3Tk} + 60R_{2Tk}R_{3Tk} + 30R_{1Tk}R_{4Tk})S_{3k} + (90R_{Tk}^{-3}R_{1Tk}^{4} \\
& - 42R_{Tk}^{-2}R_{1Tk}^{5} + 3R_{Tk}^{-1}R_{1Tk}^{6} - 180R_{Tk}^{-2}R_{1Tk}^{2}R_{2Tk} + 120R_{Tk}^{-1}R_{1Tk}^{3}R_{2Tk} \\
& - 15R_{1Tk}^{4}R_{2Tk} + 45R_{Tk}^{-1}R_{2Tk}^{2} - 90R_{1Tk}R_{2Tk}^{2} + 60R_{Tk}^{-1}R_{1Tk}R_{3Tk} \\
& - 60R_{1Tk}^{2}R_{3Tk} - 15R_{4Tk})S_{4k} + (-45R_{Tk}^{-2}R_{1Tk}^{4} + 12R_{Tk}^{-1}R_{1Tk}^{5} - R_{1Tk}^{6} \\
& + 90R_{Tk}^{-1}R_{1Tk}^{2}R_{2Tk} - 60R_{1Tk}^{3}R_{2Tk} - 45R_{2Tk}^{2} - 60R_{1Tk}R_{3Tk})S_{5k} \\
& + (40R_{Tk}^{-2}R_{1Tk}^{3} - 30R_{Tk}^{-1}R_{1Tk}^{4} + 6R_{1Tk}^{5} - 60R_{Tk}^{-1}R_{1Tk}R_{2Tk} + 90R_{1Tk}^{2}R_{2Tk} \\
& + 20R_{3Tk})S_{6k} + (-20R_{Tk}^{-1}R_{1Tk}^{3} + 15R_{1Tk}^{4} + 60R_{1Tk}R_{2Tk})S_{7k} \\
& + (15R_{Tk}^{-1}R_{1Tk}^{2} - 20R_{1Tk}^{3} - 15R_{2Tk})S_{8k} - 15R_{1T}^{2}S_{9k} + 6R_{1T}S_{10k} \\
& + S_{11k}] / S_{0k}
\end{aligned} \tag{64}
$$

where

$$
S_{Ik} = \begin{cases} \displaystyle\sum_{m=1}^{\infty}\left(\frac{\alpha_m^{I}e^{-\alpha_m^2 T_k}}{J_{1m}}J_{1mk}\right), & J_{1mk} = J_1(\alpha_m R_{Tk}), \quad I: \text{even integer} \\ \displaystyle\sum_{m=1}^{\infty}\left(\frac{\alpha_m^{I}e^{-\alpha_m^2 T_k}}{J_{1m}}J_{0mk}\right), & J_{0mk} = J_0(\alpha_m R_{Tk}), \quad I: \text{odd integer} \end{cases} \tag{65}
$$

The same procedure can be repeated to obtain the coefficients of terms with order higher than six in Eq. (51), while a lot of patience and carefulness are required.

Let ν_h = 1.6 and $T_k = 3 \times 10^{-2}$. The corresponding $R_{Tk\nu h}$ calculated numerically from Eq. (18) is 4.027017×10^{-1}, and R_{Tk} is shown in Eq. (49) as a function of ν. If ν = 1.5, 1.6 and 1.7, R_{Tk} calculated from Eq. (49) are 4.395005×10^{-1}, 4.027017×10^{-1} and 3.657039×10^{-1}, and R expressed by Eq. (51) are (Fig. 5)

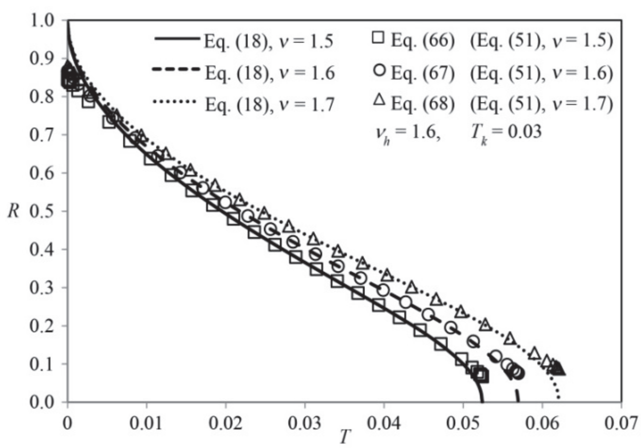

Fig. 5 R versus T in a cylinder medium

$$
\begin{aligned}
R = {} & 4.395005 \times 10^{-1} \\
& - 1.055153 \times 10\ (T - 0.03) + 4.772333 \times 10\ (T - 0.03)^2 \\
& - 1.620847 \times 10^3 (T - 0.03)^3 + 1.437144 \times 10^4 (T - 0.03)^4 \\
& - 8.821849 \times 10^5 (T - 0.03)^5 + 5.533539 \times 10^6 (T - 0.03)^6 + \cdots
\end{aligned} \tag{66}
$$

$$
\begin{aligned}
R = {} & 4.027017 \times 10^{-1} \\
& - 1.130330 \times 10\ (T - 0.03) + 4.570788 \times 10\ (T - 0.03)^2 \\
& - 1.942548 \times 10^3 (T - 0.03)^3 + 6.804401 \times 10^3 (T - 0.03)^4 \\
& + 1.976843 \times 10^3 (T - 0.03)^5 - 6.949830 \times 10^6 (T - 0.03)^6 + \cdots
\end{aligned} \tag{67}
$$

$$R = 3.657039 \times 10^{-1} - 1.208999 \times 10 \ (T-0.03) + 4.083862 \times 10 \ (T-0.03)^2 - 2.439318 \times 10^3 (T-0.03)^3 - 1.063004 \times 10^4 (T-0.03)^4 - 2.200760 \times 10^6 (T-0.03)^5 - 4.228798 \times 10^7 (T-0.03)^6 + \cdots \quad (68)$$

4. DISCUSION AND CONCLUSION

Diffusion depth, $r_a - r_f$, is very important in analyzing a diffusion case of cylinder medium, which can be calculated numerically from Eq. (18) based on given material properties, D and ν, and diffusion duration. When $t \le t_c$, the diffusion depth varies and is required in every step of calculating the average concentration of diffusing substance for characterizing diffusing properties. Calculating diffusion depth numerically needs iteration repeatedly and is much more tedious than that by a polynomial of diffusion duration. The goal of this work is to develop a process to derive the functional relationship between r_f and t to prevent the tedious numerical iteration of calculation.

Normalized location of diffusion front and normalized diffusion duration are used in this work, and the whole process has three major parts shown as a flow chart in Fig. 6.

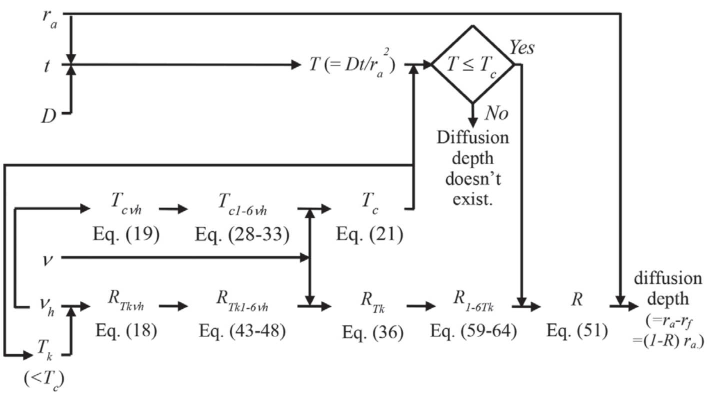

Fig. 6 Flow chart of calculating r_f from r_a, t, D, ν, ν_h, and T_k

In the first part, T_c is analyzed by Tsai process and expressed as a polynomial of ν, which is used to calculate t_c and critical for finding out whether the medium is contained thoroughly or not. Diffusion front exists only when t is smaller than t_c. This work focuses only in the period when diffusion front exists. In the second part, R is analyzed by Tsai process and expressed as a function of, when T is given as a constant, T_k. In the third part, R is analyzed by Tsai process again and expressed as a function of T. Finally, R is expressed as a polynomial of for arbitrary. Diffusion depth equals $r_a - r_f = (1 - R)\ r_a$, by which most tedious numerical iteration of calculation is prevented for calculating diffusion depth.

When t is very small, T is small, and r_f is very close to r_a. The local diffusion phenomenon happening around the surface of a cylinder medium should be very close to that of a semi-infinite medium (Tsai *et al.* 2015). As a result, the diffusion depth becomes

$$r_a - r_f = 2\nu\sqrt{Dt} \quad or \quad r_f = r_a - 2\nu\sqrt{Dt} \quad (69)$$

and

$$R = \frac{r_f}{r_a} = 1 - 2\nu\frac{\sqrt{Dt}}{r_a} = 1 - 2\nu\sqrt{T} \quad (70)$$

The slope of R with respect to T becomes

$$\frac{\partial R}{\partial T} = -\frac{\nu}{\sqrt{T}} \quad (71)$$

which approaches -∞ when T approaches 0 as shown in Fig. 5. Since

$$\frac{\partial R}{\partial T} = \frac{r_a}{D}\frac{\partial r_f}{\partial t} \quad (72)$$

the speed of r_f traveling toward the center of the medium becomes infinite at the same time.

At the same time, Eq. (59) shows that

$$\frac{\partial R}{\partial T} = R_{1T} = -S_1 / S_0 \quad (73)$$

When $T = T_c$, $R = 0$, $S_0 = 0$ and $\partial R/\partial T$ approaches $-\infty$ as shown in Fig. 5, meaning that the speed of r_f traveling toward the center of the medium becomes infinite.

Discovery of these two singularities of $\partial R/\partial T$ is novel and important for studying the efficiency of a collection tube of reverse osmosis, which is not in the scope of this work and will be in the future work.

Figure 5 shows that when $T < 0.1T_c$ or $T > 0.9T_c$, the values of R calculated based on the seven terms on the right hand side of Eq. (66), Eq. (67) or Eq. (68) deviate from those calculated numerically from Eq. (18) and the difference can be as higher as 0.149 indicating that the matching is not good when t approaches the two singularities, zero and T_c. However, in the range $0.1T_c \le T \le 0.9T_c$, the difference is less than 0.0146, and in the range $0.15T_c \le T \le 0.85T_c$, the difference is less than 0.00301. Note that the closer and T to the chosen ν_h and T_k the more accurate the calculated R is.

Better matching can be achieved if terms of order higher than six are derived and used in the Tsai process. Deriving the functions for the coefficients in Taylor series requires a lot of patience and carefulness. However, once those functions are derived, the coefficients can be calculated easily. After $T_{c\nu h}$ in Eq. (19) and $R_{\nu hTk}$ in Eq. (18) corresponding to the chosen ν_h and T_k are calculated numerically at the very beginning, no numerical iteration of calculation is needed in the following process for arbitrary ν and T.

5. NOTATIONS

C	correction factor of the advancing model
C	concentration of diffusing substance in a medium, mass of diffusing substance in unit volume of medium
$c(t;r)$	c as a function of t and r
$c(t;x)$	c as a function of t and x
c_∞	saturated c
D	diffusivity
Erf	error function
J_0 and J_1	first kind of Bessel function of order zero and order one
J_{0mR} and J_{1mR}	$J_{0mR} = J_0(\alpha_{m\,m}R)$ and $J_{1mR} = J_1(\alpha_{m\,m}R)$
J_{0mk} and J_{1mk}	$J_{0mk} = J_0(\alpha_{m\,m}R_{Tk})$ and $J_{1mk} = J_1(\alpha_{m\,m}R_{Tk})$
J_{0mkh} and J_{1mkh}	$J_{0mkh} = J_0(\alpha_{m\,m}R_{Tk\nu h})$ and $J_{1mkh} = J_1(\alpha_{m\,m}R_{Tk\nu h})$
J_{1m}	$J_{1m} = J_1(\alpha_m)$
m and n	indices of a series
R	normalized r_f, $R = r_f/r_a$
R_{1T}, R_{2T}, R_{3T}, R_{4T}, R_{5T} and R_{6T}	$R_{1T} = \partial R/\partial T$, $R_{2T} = \partial^2 R/\partial T^2$, $R_{3T} = \partial^3 R/\partial T^3$, $R_{4T} = \partial^4 R/\partial T^4$, $R_{5T} = \partial^5 R/\partial T^5$ and $R_{6T} = \partial^6 R/\partial T^6$
R_{Tk}, R_{1Tk}, R_{2Tk}, R_{3Tk}, R_{4Tk}, R_{5Tk} and R_{6Tk}	R, R_{1T}, R_{2T}, R_{3T}, R_{4T}, R_{5T} and R_{6T}, at $T = T_k$.
$R_{Tk1\nu}$, $R_{Tk2\nu}$, $R_{Tk3\nu}$, $R_{Tk4\nu}$, $R_{Tk5\nu}$ and $R_{Tk6\nu}$	$\partial R_{Tk}/\partial\nu$, $\partial^2 R_{Tk}/\partial\nu^2$, $\partial^3 R_{Tk}/\partial\nu^3$, $\partial^4 R_{Tk}/\partial\nu^4$, $\partial^5 R_{Tk}/\partial\nu^5$ and $\partial^6 R_{Tk}/\partial$
$R_{Tk\nu h}$, $R_{Tk1\nu h}$, $R_{Tk2\nu h}$, $R_{Tk3\nu h}$, $R_{Tk4\nu h}$, $R_{Tk5\nu h}$ and $R_{Tk6\nu h}$	R_{Tk}, R_{Tk1}, $R_{Tk2\nu}$, $R_{Tk3\nu}$, $R_{Tk4\nu}$, R_{Tk5} . and $R_{Tk6\nu}$ at $\nu = \nu_h$
R	radian coordinate of polar coordinates system
r_a	radius of a solid cylinder medium
r_f	location of diffusion front in a cylinder medium for polar coordinates system
S_I	$S_I = \begin{cases} \sum_{m=1}^{\infty}\left(\frac{\alpha_m^I e^{-\alpha_m^2 T}}{J_{1m}} J_{1mR}\right), \frac{\partial S_I}{\partial T} = -S_{I+2} + R_{1T} S_{I+1} - R^{-1} R_{1T} S_I, \ I: \text{even integer} \\ \sum_{m=1}^{\infty}\left(\frac{\alpha_m^I e^{-\alpha_m^2 T}}{J_{1m}} J_{0mR}\right), \frac{\partial S_I}{\partial T} = -S_{I+2} - R_{1T} S_{I+1}, \ I: \text{odd integer} \end{cases}$
S_{Ik}	$S_{Ik} = \begin{cases} \sum_{m=1}^{\infty}\left(\frac{\alpha_m^I e^{-\alpha_m^2 T_k}}{J_{1m}} J_{1mk}\right), \quad J_{1mk} = J_1(\alpha_m R_{Tk}), \quad I: \text{even integer} \\ \sum_{m=1}^{\infty}\left(\frac{\alpha_m^I e^{-\alpha_m^2 T_k}}{J_{1m}} J_{0mk}\right), \quad J_{0mk} = J_0(\alpha_m R_{Tk}), \quad I: \text{odd integer} \end{cases}$
T	normalized t, $T = Dt / r_a^2$
T_c	$T_c = Dt_c / r_a^2$
$T_{c1\nu}$, $T_{c2\nu}$, $T_{c3\nu}$, $T_{c4\nu}$, $T_{c5\nu}$ and $T_{c6\nu}$	$T_{c1\nu} = dT_c/d\nu$, $T_{c2\nu} = d^2T_c/d\nu^2$, $T_{c3\nu} = d^3T_c/d\nu^3$, $T_{c4\nu} = d^4T_c/d\nu^4$, $T_{c5\nu} = d^5T_c/d\nu^5$ and $T_{c6\nu} = d^6T_c/d\nu^6$
$T_{c\nu h}$, $T_{c1\nu h}$, $T_{c2\nu h}$, $T_{c3\nu h}$, $T_{c4\nu h}$, $T_{c5\nu h}$ and $T_{c6\nu h}$	T_c, $T_{c1\nu}$, $T_{c2\nu}$, $T_{c3\nu}$, $T_{c4\nu}$, $T_{c5\nu}$ and $T_{c6\nu}$ at $= \nu_h$.
T_k	a chosen value for T
t	diffusion duration
t_c	the critical diffusion duration when the diffusion front reaches the center of a cylinder medium
x, y and z	coordinates of Cartesian coordinate system

z	also for axial coordinate of polar coordinates system
α_m	m-th zero of J_0
ν	Neumann's constant
ν_h	a chosen value for ν
θ	angular coordinate of polar coordinates system

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