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Investigating Differential Difference Equations: An In-Depth Review

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ABSTRACT

Differential difference equations serve as mathematical models for a wide range of physical phenomena. Differential difference equations have a broad spectrum of applications spanning various disciplines. These equations find diverse applications in a wide range of fields, including epidemiology, information technology, control theory, finance, population dynamics, and stochastic processes. Their adaptability in modeling systems characterized by a blend of continuous and discrete behaviors renders them an invaluable mathematical framework with applicability spanning numerous domains in science and engineering. This study provides a comprehensive review of existing literature pertaining to periodic solutions, entire solutions, asymptotic analysis, and numerical approaches for differential difference equations. The pri- mary objective of this research is to analyze the various problems and methodologies employed in the extant literature. Notably, significant advancements in this field have been made since 1946. As a result, this study aims to encompass the research conducted by various scholars from 1946 to 2023.

*keywords:*Differential difference equation; Numerical approaches; Periodic solutions; Entire solutions; Asymptotic analysis.

1. INTRODUCTION

Differential-difference equations (DDEs) are crucial in the modeling of various physical phenom- ena, including electrical network currents, vibrations in particle lattices, and pulses in biological chains, as cited in reference [1]. While difference equations involve complete discretization, DDEs are characterized by a partial discretization of some or all of their variables, while typi- cally maintaining continuous time. Differential equations are a common approach for describing a wide range of physical problems. However, when dealing with situations where space or time exhibits discontinuities, the conventional differential model may no longer be applicable. El- Naschie's E-infinity theory suggests that both space and time can exhibit discontinuities. While time can often be approximated as continuous in many practical applications, at the nanoscale and even smaller scales, numerous problems exhibit discontinuities. In these scenarios, the use of differential-difference models proves to be highly effective, as highlighted in references [2] and [3]. Furthermore, additional applications of DDEs can be found in fields such as textile en- gineering and the study of stratified hydrostatic flows, as demonstrated in references [4] and [5]. Other applications of DDEs and their systems appear in many real life phenomenon in control theory, engineering, physics, economics, engineering, astronomy, chemistry, mechanics, biology, electrostatics, potential theory etc., [6–18].

Therefore, many researchers always curious to determine the solution of DDEs. This review will explore the endeavors of various authors who have studied into the realm of solving DDEs. Their efforts span both the analytical and numerical approaches, and we will examine how different researchers, across different years, have made significant contributions to this field by seeking solutions to DDEs through various methods.

The models of DDEs considered in this review shown in the flow diagram in Figure 1.

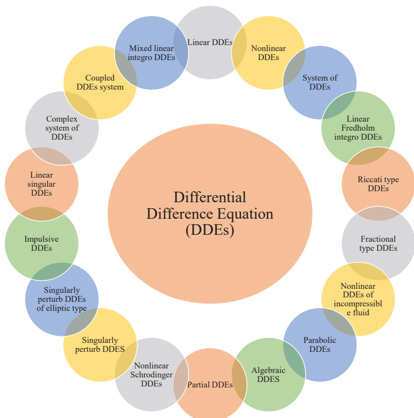

Fig. 1 The review specifically analyzed different models of differential difference equations

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2. MODELS OF DIFFERENTIAL DIFFERENCE EQUATIONS

In [19], Cooke discussed the presence and consistency of periodic solution for the following equation:

$$\frac{d}{ds}y(s+1) = ay(s) + by(s+1) + g(y(s), y(s+1), \nu s) + nf(\nu s). \quad (1)$$

We assume that the following conditions are satisfied:
(i) All of the characteristic equation's roots

$$re^r - be^r - a = 0, \quad (2)$$

has real negative elements.

(ii)*a, b, n,* and *v* represent real numbers with *v* being greater than 0, while s stands as a real variable, and s is a real.

(iii)The function *f(s)* is a real and continuous function, defined for all values of s, and it exhibits a periodicity of 1, with its mean value over one period being equal to zero.

(iv)The function g*(x, y, s)* is real and continuous for all values of *(x, y, s)* within a period of 1 in the variable s, and it also satisfies the condition *g*(0, 0, *s*) = 0. Furthermore, for each ζ > 0

there corresponds a positive number△, depending on ζ but not on s such that

$$|g(x_1, y_1, s) - f(x_2, y_2, s)| \leq \zeta(|x_1 - x_2| + |y_1 - y_2|), \quad (3)$$

∀ real s and ∀ x_1, x_2, y_1, y_2 for each $|x_1 - x_2| < \triangle$ and $|y_1 - y_2| < \triangle$.

Cooke inaugurated the theorem with the statement as:

Theorem: If the above stated conditions are fulfilled, there ∃ a +ve constant η, depending on a, b, g, and f but not on n or v, such that if, $\frac{|k|}{(1+\nu)} < \eta$*, then equation eq. (1) posses a continuous,*

periodic solution p(s) of period $\frac{1}{\nu}$*. Furthermore, ∃ a +ve constant ϱ that does not depend on either n or v, and this constant satisfies the following conditions:*

$$|p(s)| \leq \frac{\varrho|n|}{1+\nu}. \quad (4)$$

∀s. The solution p(s) is asymptotically stable and in fact, ∃ a +ve number δ that is independent of both n and v. This δ ensures that any solution x(s) of eq. (1) which adheres to the following conditions:

$$\max_{0\leq s<1} |x(s)| \leq \delta, \quad (5)$$

will also satisfy

$$\lim_{t\to\infty} [x(s) - p(s)] = 0. \quad (6)$$

In [20], the work of Farnell et al. on the system of differential equations directed towards the DDEs by the above stated theorem. Bellman [21], Cooke [22] , Brownell [23], and Wright in several publications, including one cited as [24], conducted extensive investigations into the stability and asymptotic characteristics of solutions for DDEs similar to equation (3), with the distinction of lacking a forcing term. The methods practiced in [19] are taken primarily from [20] and [21]. In [25], Jones consider the non-linear DDE as:

$$\acute{g}(t) = -\beta g(t-1)\{1 + g(t)\}, \quad t > 0. \quad (7)$$

If we assume *β* is a positive real parameter and let *ψ* be a bounded, real-valued, a Lebesgue- integrable functional specified on the range of values (-1, 0] and the function g in the following manner: for values of t within the interval (-1, 0], g*(t)* is equal to *ψ*(t), and for other values of *t*,

$$g(t) = g(0) - \alpha \int_0^t g(x-1)\{1 + g(x)\}dx, \quad (8)$$

for t > 0 is specified as the solution of (7) related to the initial function *ψ*. eq. (7) finds relevance in diverse applications, such as modeling fluctuating populations of organisms, as referenced in [26]. As noted in [27], it also occurs in the context of probability techniques for prime number distribution analysis. Moreover, as noted in [28], this equation is found in the context of demonstrating control systems, and analogous equations are employed in business cycle-focused economic research.

In [29], Wright has determined the presence and uniqueness of the solution of eq. (7) related to each initial function and presented the equation

$$g(t) = \{1 + g(0)\}exp\{-\beta \int_{-1}^{t-1} \{g(x)dt\} - 1, \quad (9)$$

from where g can be subsequently determined over unit intervals. He also established the trivial solution's asymptotic stability in interval 0 ≤ *β* ≤ 3/2, the persistence of constrained

oscillatory solutions that are undamped for *β*=>(*r*/2) and many other stimulating results together with various extensions for solutions. In [30], Kakutani and Markus demonstrated the theories isolating solutions' stiff asymptotic behaviour for α in (0, e^{-1}) and the apparent oscillating behaviour for a > e^{-1}. Moreover, the author's contribution in [25] is regarded with the undamped oscillatory solutions of eq. (7) which happen for *β* > ($\frac{\pi}{2}$).The author showcased the existence of periodic solutions and elucidated several of their properties by presenting refined constraints on amplitudes and outlining limitations on the behavior of these solutions. He also computed numerical results by using a computer, which proved the accuracy of solutions for a fixed periodic form of *β* > ($\frac{\pi}{2}$).

In [31], Brauer consider the linear DDE with constant coefficient as:

$$\acute{x}(t) = ax(t) + bx(t-T), \quad (10)$$

and b≠ 0 for the explicit dependence on delay *T*.

eq.(10) has a characteristic equation having all roots with negative real parts.The characteristic equation, which is derived from the coefficients of the differential difference equation and incorporates the delay parameter *T*, plays a crucial role. The roots of this characteristic equation are utilized to determine the asymptotic behavior of the solutions. This research holds valuable significance in precisely studying the behavior of solutions for nonlinear first-order DDEs in the vicinity of a stable equilibrium. The applications of this research find relevance in addressing control and biological problems.

In [32, 33] Brayton studied the neutral DDE as:

$$\dot{x}(t) - i\dot{x}(t - \frac{2}{r}) = g(x(t), x(t - \frac{2}{r}), \quad (11)$$

where $r = \sqrt{LC}$, eq. (11) based on the lossless broadcasting lines which are associated with switching circuits. Different papers, such as those referenced in [34] and [35], have presented the necessary conditions for the presence of periodic solutions in equation (11), specifically when *i* = 0. But only a few papers for the cases $k \neq 0$ exist in the literature that investigates whether periodic solutions exist of eq. (11). In [36], Chen consider the more general form of the class of neutral DDE than (11) and investigated if the equation below has any periodic solutions:

$$(y(t) + ay(t-\tau))' = -g(y(t), y(t-\tau)). \quad (12)$$

For a positive value of τ, a real number denoted as a, and a continuous function represented as *g(x, y)*

In [37], Stamov and Stamova consider the impulsive DDE:

$$\begin{aligned} y(s) = f(s, y(s), y(s-h)), \quad s > s_0, \quad s \neq \tau_l(y(s)), \\ y(s) = \phi_0(s), \quad t\epsilon[s_0 - h, s_0], \\ \Delta y(s) = I_l y(s), \quad s = \tau_l(y(s)), \\ s > s_0, \quad l = 1, 2, 3, ..., \end{aligned} \quad (13)$$

where R^m, be the m-dimensional Euclidean space with norm $|y| = (\sum_{l=1}^{m} y_l^2)^{\frac{1}{2}}$, let Ω be domain in R^m, $\Omega \neq \phi$, h > 0, $s_0 \in$ R, $\phi_0 \in$ C [[$s_0 - h$, s_0], Ω] , $R_+ = [0, \infty)$. Where f : $(s_0, \infty)\times\Omega\times\Omega \rightarrow R^m$, $\tau l : \Omega \rightarrow (s_0, \infty)$, $I_l : \Omega \rightarrow R^m$, l = 1, 2, ...,$\Delta z(s) = z(s + 0) - z(s - 0)$. In this study, the author established the necessary conditions for the presence of integral manifolds in impulsive DDEs that incorporate variable impulsive perturbations. This was achieved by using auxiliary function that are component-wise continuous and comparable to the conventional Lyapunov's function.

In [38], Chou et al. consider the DDE of the form:

$$\begin{aligned} v_s(y) = I(s, y, v(y-ch), ..., v(y+dh)) \equiv I(s, y, v, h), \\ t \in R^+, \quad ,y \in Z_h, \quad ,c, d \in Z^+, \end{aligned} \quad (14)$$

where *s* represents the continuous time variable, and *v* is a differential function with respect to time variable s Additionally, Z_h is defined as the infinite uniform spatial grid with grid points at intervals of h such that $Z_h = y = ih$, $i = 0, \pm1, \pm2, \ldots$, where h is a very small step size. Lastly, *I* is a differential function of various variables, including *s*, *y*, *v* (*y*−*ch*), *v* (*y*+*dh*). The authors have introduced a generalized conditional symmetry method in their study to handle nonlinear DDEs that encompass continuous dependent variables and are relevant for both continuous and discrete independent variables. Furthermore, they extended the application of the generalized conditional symmetry method from nonlinear partial differential equations to DDEs. As a result of their efforts, they successfully derived exact solutions of DDEs.

In [39], Suzuki studied retarded system of nonlinear DDEs for boundary value problem as:

$$\dot{y}(r) = G(r, y(r), y(r-l_1), ..., y(r-l_i) \qquad (0 < r \leq m), \quad (15)$$

$$y(\phi) = Ny(m+\phi) + \theta(\phi) \qquad (-l \leq \phi \leq 0), \quad (16)$$

where l_1, ...l_i and *m* have positive values that satisfied $l := max l_1, ..., l_i < m$

In [40], Urabe determined the existence theorem of multipoint nonlinear ODEs boundary value

problems and presented a method for estimating an exact solution by employing chebyshev polynomials. In [39], the author extended the work of M. Urabe [40] to nonlinear DDE (15) and (16). The author proved the 'theorem-1' on the based of the results proved by one of the researchers Kurihara and Yui [41]. It delivered that the presence of the unique exact solution could be always guaranteed toward the boundary value problems (15) and (16) that satisfied the isolated conditions through examining different hypotheses against obtained approximate solutions. Moreover, the author proved 'theorem-2' which delivered that the desired precison of approximate solutions can be determined by calculating the finite chebyshev polynomial series for exact solution toward the boundary value problems (15) and (16) that satisfied the isolated conditions.

In [42], Muravnik consider the parabolic DDEs of Cauchy problem in $R^1 \times (0, \infty)$:

$$\frac{\partial v}{\partial x} = \frac{\partial^2 v}{\partial y^2} + \sum_{l=1}^{n} c_l v(x - h_l, x), \quad (17)$$

where *c*, *h* $\in R^n$. The author constructed the fundamental solution as well as integral description of the classical solution of problem (17).

In [43], Kadalbajoo and Sharma studied the singularly perturb DDE as:

$$\epsilon^2 z''(r) + \beta(r)z(r-\eta) + y(r)z(r) + \gamma(r)z(r+\delta) = g(r), \quad (18)$$

on [0, 1] with boundary conditions

$$\begin{aligned} z(r) = \psi(r), \quad -\eta \leq r \leq 0, \\ z(r) = \phi(r), \quad 1 \leq r \leq 1+\delta. \end{aligned} \quad (19)$$

In reference [44], the focus is on the numerical handling of 2nd order singularly perturbed DDEs having small shifts. In this context, ϵ represents a small parameter, and it is required that $0 < \epsilon \ll 1$. Additionally, there are small shifting parameters η and δ, with the conditions that $0 < \eta \ll 1$ and $0 < \delta \ll 1$. And $\beta(x)$, $\gamma(x)$, $g(x)$, $\psi(x)$, and $\phi(x)$ are all smooth functions involved in the analysis. In [43] K. K. Sharma and M. K. Kadalbajoo prolonged the work to the problems having solution with rapid oscillations. They used the Taylor series to approximate the shifted term and then employ the difference scheme, yielded singular perturbation parameters of small shifts. They also discussed the consistency and convergence of the technique. The impact of small changes of the oscillatory solution was demonstrated using a variety of numerical examples. In [45] Sezer and Gulsu gave Taylor matrix method of higher order general Fredholm integro linear DDE

$$\sum_{l=0}^{n}\sum_{i=0}^{q} P_{li}(r)^z(l)(r-\tau_{li}) = g(r) + \int_a^b \sum_{j=0}^{p}\sum_{k=0}^{s} K_{jk}(r,t)z^{(j)}(t-\tau_{jk})dt,$$
$$\tau_{li} \geq 0, \tau_{jk} \geq 0, \quad (20)$$

having mixed conditions

$$\sum_{l=0}^{n-1}\sum_{s=1}^{S} c_{kl}^{s} z^{(l)}(c_s) = \lambda_k, \quad l = 1, 2, 3, ..., m, \quad a \leq c_s \leq b, \quad (21)$$

the solution is demonstrated in Taylor polynomial as

$$y(x) = \sum_{m=0}^{M} \frac{z^m(c)}{m!}(x-c)^m, \quad a \leq x, \quad c \leq b. \quad (22)$$

Where, the functions $P_{li}(r)$, $K_{jk}(r, t)$, and $g(r)$ that posses $a \leq r, t \leq b$ and c_{kl}^{s}, c_s, c, and τ_{li}, τ_{jk} are appropriate coefficients and $y^m(c)$ are coefficients of Taylor to be resolved. The authors de- vised the Taylor matrix method and employed this technique to provide an approximate solution for the aforementioned equations, particularly when they involve mixed boundary conditions.

They used Taylor polynomials as a key component of their approach. The accuracy of the pro- posed method is validated through some numerical examples solved by using MAPLE. They calculated absolute error to prove the accuracy.

In [46] Kadalbajoo and Sharma provided a numerical solution for singularly perturbed nonlinear DDEs featuring a negative shift

$$\epsilon u''(x) = G\left(x, u(x), u'(x-\eta)\right). \tag{23}$$

This will format the equation properly and label it as (23). on (0, a) based on the boundary conditions

$$u(x) = \psi(x), -\eta \le x \le 0, \; y(a) = \alpha, \tag{24}$$

where $0 < \epsilon \ll 1$ and η is the shift. They presented the numerical treatment for eq. (23) and eq. (24). The solution technique is divided into two sections based on the size of the shift. They employed Taylor series to manage the term related to $o(\epsilon)$ and devised a specific mesh technique to address the term associated with $o(\epsilon)$.

In reference [47], Patidar and Sharma studied a numerical approach for handling singularly perturbed DDEs that incorporate both delay/advance arguments, much like the approach de- scribed in reference [43]. They introduced a novel category of operator finite difference schemes through the utilization of nonstandard finite difference methods. They presented noteworthy findings concerning problems with constant or variable coefficients, even when the parameters η and δ are set at $\eta = \delta = 0.5\epsilon$. Furthermore, they demonstrated that these nonstandard methods consistently produced $\epsilon - uniform$ results across a range of parameter values for η and δ.

In [48] Arikoglu and Ozkol consider the DDE in general form

$$g[f^{(k_1)}(x+a_1), f^{(k_2)}(x+a_2), \ldots.., f^{(k_j)}(x+a_j)] = d, \tag{25}$$

under the conditions

$$[\frac{d^{\alpha_i} f}{dx^{\alpha_i}}]_{x=\beta_i} = \alpha_i, \quad fori = 1, 2, 3, \ldots., n. \tag{26}$$

The authors gave the solution of DDE by establishing new theorems by differential transform method. The general form of these theorems covered an extensive range of DDEs, either linear and nonlinear as well as with constant and variable coefficients. They transformed eq. (25) and eq. (26) into a recurrence equation, which was subsequently employed to determine the solutions of an algebraic systems using the coefficients of the power series solutions. To illustrate the effectiveness and resilience of their method, as well as to demonstrate the relevant theorems, they provided examples.

In [49] Gülsu and Sezer consider linear DDE with variable coefficients that involving negative shifts in the derivative term

$$\sum_{m=0}^{n} P_{(m)}(z)u^{(m)}(z) + \sum_{q=0}^{Q} P^*_{(q)}(z)u^{(q)}(z-\tau) = Q \le n, \tau > 0, -\tau \le z \le 0,$$

under mixed type conditions

$$\sum_{m=0}^{n-1} [p_{im}u^{(m)}(p) + s_{im}u^{(m)}(s) + d_{im}u^{(m)}(d)] = \nu_i, \tag{27}$$

$i = 0(1)(m-1)$, $p \le d \le s$ and the way in which the solution is demonstrated

$$u(x) = \sum_{k=0}^{K} \frac{u^{(k)}d}{k!}(z-d)^{(k)}, p \le z \le s, K \ge n, \tag{28}$$

The expression refers to determining coefficients, denoted as $u^{(k)} d$ for $k = 0, 1, \ldots, K$, in a Taylor polynomial of degree K centered at the point $z = d$. The functions $P_{(m)}(z)$, $P^*(q)(z)$, and $g(z)$ are defined within the interval $p \le z \le s$, and the constants p_{im}, s_{im}, d_{im}, and v_i are real coefficients associated with the problem at hand. In reference [50], Fredholm integral equations were successfully addressed using the Taylor method. Furthermore, Sezer extended this method to tackle a wider range of problems, including Fredholm integro-differential equations' solution (as seen in reference [51]) and 2nd order linear differential equations (refer to references [52] and [53]). In another work, reference [49], the authors introduced the Taylor polynomial method, which involves employing trimmed Taylor expansions for the functions within the context of DDEs and subsequently incorporating their matrix representations into the equations. Therefore, the unknown Taylor coefficients might be located by solving the resulted matrix equation.

In [54], Sezer and Dascioglu consider linear DDE with variable coefficients

$$\sum_{m=0}^{n}\sum_{i=0}^{p} P_{(im)}(x)u^{(m)}(x-\tau_{im}) = g(x), \tau_{im} \ge 0, \tag{29}$$

under mixed conditions

$$\sum_{m=0}^{n-1}\sum_{q=1}^{Q} a^q_{jk}u^{(k)}(a_q)] = \nu_j, \tag{30}$$

and solution is expressed as,

$$u(x) = \sum_{m=0}^{M} s_m(x-a)^m, s \le x, a \le d, \tag{31}$$

$$s_n = \frac{u^{(m)}a}{m!}, m = 0, 1, 2, \ldots, M, \tag{32}$$

here $P_{(im)}(x)$ and $g(x)$ containing appropriate derivatives on $s \le x \le d$ and a^q_{jk}, a_q, a and τ_{im} are appropriate coefficients. The authors are constructed Taylor method and gave the solution of eqs. (29) and (30) by using this method. They also demonstrated the admissible characteristic of the method by presenting different numerical examples.

In [55], Zonghang Yang and Y. C. Hon gave exact travelling wave solution of DDE by using hyperbolic cotangent function method. By extending the hyperbolic cotangent function method, they enriched the range of exact solutions available for nonlinear DDEs, introducing distinct variations. The intended method is tested by finding the solution of three different types of DDEs, the mKdV lattice, Toda lattice and the Volterra lattice in modified form that have been discretized. Moreover, they used correlation features of hyperbolic functions and triangular functions for interpreting periodic wave solutions of triangular type.

In [56], Wang, Zoub and Zhang studied the Volterra equation

$$\frac{\partial z_n}{\partial x} = z_n(z_{n+1} - z_{n-1}), \tag{33}$$

along initial condition as

$$z_n(0) = n, \tag{34}$$

contains the exact solutions

$$z_n(x) = \frac{n}{1-2x}, \tag{35}$$

the homotopy analysis method, initially limited to integral differential equations, has been expanded by the authors to address non-linear DDEs. They utilized this extended method to tackle equations (33) and (34), demonstrating its effectiveness and significant potential. Additionally, they established a convergence theorem and provided a succinct analysis of the results obtained.

In [57] Li et. al consider the Toda equation of (2+1) -dimensions

$$z_{n,xt} = e^{z_{n-1}-z_n} - e^{z_n - z_{n+1}}, \tag{36}$$

here function called z_n depends on x and t, and $zn,xt \equiv$

Based on the geometric composition of differential equations by Cartan, Estabrook and Harrison proposed a geometric approach for the symmetry of differential equations. The authors interpreted the DDE sym- metries and expanded the research of Estabrook and Harrison. The Lie symmetry properties of the (2+1)-dimensional Toda equation were examined using the discrete exterior differential technique.

In [58], Ma et al. formulated symmetrical Fibonacci tane based on the sine and cosine symmetric Fibonacci. They created an approach to obtain the exact travelling waveform solution of the DDE by utilising the symmetrical Fibonacci tane functions feature. They developed remarkable explicit as well as exact traveling wave solutions by employing this method on discrete non-linear Schrodinger problem, Toda lattice of (2+1) dimensions and generalized Toda lattice.

In [59], Kadalbajoo and Kumar consider the singularly perturbed DDE having negative shift in its first derivative

$$\zeta u^{''} + p(z)u^{'}(z-\eta) + q(z)u(z) = r(z), \tag{37}$$

along with boundary conditions

$$u(y) = \phi(y), \quad for \ -\eta \leq y \leq 0, \quad u(1) = \beta. \tag{38}$$

Here 'ζ' is a small argument with the condition that $0 < \zeta \ll 1$, and η is a small shifting parameter,fulfilling the requirements that $0 < \eta \ll 1$. The functions $p(y)$, $q(y)$, $r(y)$, $\phi(y)$ are all smooth, and the constant β is also part of the equation. They employed a fitted mesh approach to create a segmented mesh that maintains uniformity while being compressed in the vicinity of the boundary layers. In conjunction with the fitted mesh, they utilized the B-spline collocation approach.An approximate 2nd order parameter homogeneous convergence was shown by this method. The effect of a minor delay η upon the boundary layer was also covered by the authors. Furthermore, they validated the method's effectiveness through various illustrative examples. These examples also showcased how the delay's argument size and the delay term's coefficient influence the layer behavior of the proposed solution.

In [60], Zou, Wang and Zong consider the algebraic DDE in general form

$$N(u_n(t), u_{n+1}(t), u_{n-1}(t), u_{n+2}(t), u_{n-2}(t), ...) = 0. \tag{39}$$

Here, non-linear differential operator is represented by N, and both 'n' and 't' serve as independent variables. The functions $u_n(t)$ are represented in vector form. The authors introduced a novel method, the differential transform Padé approximation method, for solving DDEs. They extended the traditional differential transform method and incorporated the Padé approach. By combining these techniques, they aimed to broaden the convergence region of the series solutions. Their approach yielded successful results, particularly in obtaining solitary-wave solutions for the discrete KdV and mKdv equations. A comparison between the results derived from their method and the exact solutions demonstrated the robustness and reliability of their approach. In [61], Zhang et. al consider a system of N polynomial of non-linear DDEs

$$\Delta(y_{n+r_1}(l), ..., y_{n+r_k}(l), y^{'}_{n+r_1}(l), ..., y^{'}_{n+r_k}(l), ..., y^{(s)}_{n+r_1}(l), ..., y^{(s)}_{n+r_k}(l)) = 0, \tag{40}$$

Zhang et. al [62, 63] introduced a generalize $(\frac{G^{'}}{G})$ expansion approach to promote and prolong the findings of Wang et al. [64] for the solution of variable coefficient equations as well as high dimensional equations. The authors in this study, prolonged the $(\frac{G^{'}}{G})$ expansion approach to handle non-linear DDEs. They demonstrated the efficacy and benefits of the method by taking two discrete non-linear lattice equations using symbolic computation, obtained trigonometric and, hyperbolic function solutions as a consequence. some obvious solutions along with singular traveling wave and kink solitary wave solutions are retrieved, even when the parameters are selected as special values.

In [65], Liu consider the lattice equation in general form

$$M(z^{'}_n(t), z^{''}_n(t), ..., z^{(r_1)}_n(t), z_{n+k_1}(t), ..., z_{n+k_{r_2}}(t)) = 0. \tag{41}$$

The author developed exponential functions rational expansion scheme and gave exact traveling wave solution of non-linear DDEs. He merged the hyperbolic tangent method and made its deducement. He achieved several exact solutions by applying this method to non-linear DDEs like discrete mKdV lattice equation, Langmiuir lattice and Hybrid lattice equation.

In [66], Gulsu, ozturk and Sezer consider the mixed linear integro DDE

$$\sum_{l=0}^{n} P_l(s)y^l(s) + \sum_{r=0}^{n} P^*_{(r)}(s)Y^{(r)}(s-\xi) = f(s) + \int_{-1}^{1} K(s,t)y(t)dt, -\xi \leq s \leq 0, \tag{42}$$

under the conditions

$$\sum_{l=0}^{n-1}[a_{il}y^{(l)}(a) + b_{il}y^{(l)}(b) + c_{il}y^{(l)}(c)] = \nu_i, \tag{43}$$

here, $y(s)$ represents an unknown function, while $P_l(s)$, P^*r, $f(s)$, $K(s, t)$ are known functions specified on a specific interval. Additionally, there are constants denoted as ail, b_{il}, c_{il}, and v that are appropriately chosen. The solutions are presented in the following form:

$$y(s) = \sum_{j=0}^{M} a_j T_j(s), \tag{44}$$

In the given equation, the coefficients a_j are considered as unknown Chebyshev coefficients, and a positive integer M is chosen such that it satisfies the condition $M \geq n$. The authors developed a Chebyshev collocation technique that allows mixed linear integro-

DDEs to be solved numerically. The primary premise of this approach is the application of the Chebyshev expansion technique. By applying this approach, the mixed linear integro-DDEs, along with their associ- ated conditions, are transformed into a matrix equation, which is then equivalently expressed as linear algebraic systems. To validate the efficiency and reliability of their proposed approach, the authors conducted various numerical examples. These examples were executed using the computer algebraic system Maple 10.

In [67], Abbas Saadatmandia and Mehdi Dehghan consider the linear Fredholm integro DDE

$$\sum_{m=0}^{q} P_m(r)y^m(r) + \sum_{n=0}^{t} P^*_{(n)}(r)Y^{(n)}(r-\rho) = g(r) + \int_k^l K(r,t)(t-\rho)dt, \rho \geq 0, \quad (45)$$

along with the conditions

$$\sum_{m=0}^{q-1} [\alpha_{im} y^{(m)}(k) + \beta_{im} y^{(m)}(l) + \gamma_{im} y^{(m)}(\eta)] = \nu_i, i = 0, 1, ..., q-1, \quad (46)$$

the functions $P_m(r)$, $P^*(n)(r)$, $K(r, t)$, and $g(r)$ are functions that are continous. Additionally, there are constants denoted as αim, β_{im}, γ_{im}, v_i that are appropriately selected. The problem's spatial domain contains the point η. A higher order linear Fredholm integro-DDE was solved by the authors using Legendre polynomials. They extended the approximate solution using shifted Legendre polynomials along with unknown coefficients, by converting the problem into a set of linear equations. Subsequently, They used the tau approach in combination with operational matrices of derivative and delay to calculate the unknown coefficients related to shifted Leg- endre polynomials. A key component of their methodology was the tau technique, which was first developed by Lanczos [68] for ODEs and then expanded upon by Ortiz [69]. For more information on the tau method, references [70] and [71] are available, and for a deeper under- standing of Legendre polynomials, one can refer to [74] and [75]. To validate the accuracy of their proposed technique, they provided several examples and compared their results with those already established in the existing literature.

In [76], Hongfei Li, Keqin Gub gave discretized Lyapuno Krasovskii functional approach for linear system having multiple delays

$$\dot{y}(t) = By(t) + \sum_{i=0}^{k} C_i x_i(t - l_i), \quad (47)$$

$$x_j(t) = D_j y(t) + \sum_{i=0}^{k} E_{ij} y_i(t - l_i), j = 1, 2, ..., k, \quad (48)$$

here $y(t) \in R^n$, $x_j(t) \in R^{m_i}$. The delays l_i are all positive. Initial history might be expressed for any $t_0 \in R$,

$$y_{t_0} = \psi, \quad (49)$$

$$x_{i(s_i)t_0} = \phi_i, i = 1, 2, ..., k, \quad (50)$$

for $\psi \in R^n$ and $\phi_i \in PC(l_i, m_i)$. The current system's state at time t can be represented as $(y(t), x1(s1)t, x2(s2)$t,......, $xk(sk) t)$ from which the system's future expansion can be entirely determined. To address this, a discretized Lyapunov Krasovskii functional approach was presented by the authors.. This method yields stability conditions based on linear matrix inequalities for solving coupled DDEs with multiple delays. Further, this framework is also appropriate for the retarded type of time delay systems and remarkably reduced the cost of computations for a typical system.

In [77], Wang, Zou, and Zong initially employed the Adomian decomposition technique and Pade approximation for solving DDEs. They achieved an improved approximation by combining these two techniques. The method's convergence was established by employing it to solve both the Volterra equation and the discrete mKdV equation. Furthermore, the results obtained using their proposed method were compared with exact solutions to validate their approach.

In [78], Kumar and Kadalbajoo presented a numerical treatment for solving the singularly per- turbed DDEs with advanced and delay terms having boundary layer

$$\frac{\partial y}{\partial r} - \epsilon^2 \frac{\partial^2 y}{\partial l^2} + c(l)\frac{\partial y}{\partial l} + \mu(l)y(l-\eta, r) + \nu(l)y(l, r) + \rho(l)y(l+\delta, r) = g(l, r), \quad (51)$$

where $K = \rho\times\sigma = (0, 1)\times(0, R]$ and $\partial K = \overline{K} - K = \{(0, r)\cup(1, r)\cup(x, 0) : 0 \leq l \leq 1, 0 \leq r \leq R\}$, in the plane of space time, for constant positive time R, associated with the interval conditions

$$y(l, r) = \psi(l, r), (l, r) \in K_- = \{(l, r) : -\eta \leq l \leq 0; r \in \Lambda\}, \quad (52)$$

$$y(l, r) = \phi(l, r), (l, r) \in K_+ = \{(l, r) : 1 \leq l \leq 1 + \delta; r \in \Lambda\}, \quad (53)$$

and initial history $y(l, 0) = y0(l)$, $l \in \overline{\Omega}$. Here 'ξ' is small argument and $0 < \xi \ll 1$, and η, δ are small arguments of order $o(\xi)$. While the functions $c(l)$, $\mu(l)$, $v(l)$, $\rho(l)$, $g(l, r)$, $\psi(l, r)$, $\phi(l, r)$, $y0(l)$ are independent of ξ and supposed to be smooth and bounded. They also considered the condition for some constant θ, $\mu(l) + v(l) + \rho(l) \geq \theta > 0$, $\forall$ $1 \in \overline{\sigma}$. The eq. (51) diminished to one parameter singularly perturbed differential equation for the case $\eta = \delta = 0$. However, they were interested in two parameter problem, where the reaction terms include both the delay and the advance period.. The eq. (51) diminished to a singularly perturbed parabolic differential for the case $\eta = \delta = 0$. The boundary layer depend upon the sign of $c(l)$, for $c(l) < 0$ and $c(l) > 0$ the boundary layer that exists in the neighborhood of D_0 and D_+ respectively [79, 80]. They used Taylor's series expansion for approximating the retarded factors and obtained singularly perturbed time dependent differential equation. They employed Rothe's scheme and, the B spline collocation scheme derived from a Shishkin-type approach in the direction of space. This combination was utilized to approximate the singularly perturbed DDE. Using the standard implicitly finite difference method, parameters uniform numerical techniques were created within the framework of the approach. They exhibited the order of accuracy of the technique $O(L^{-1} + M^{-2} ln^3 M)$, and M represents the mesh points inside the spatial direction, whereas L denotes the mesh points in the temporal direction. To validate the proposed scheme, they demonstrated its consistent convergence by analyzing the numerical results. Additionally, they conducted a comparison with both an upwind finite difference scheme and a midpoint upwind finite difference scheme, utilizing a piecewise uniform mesh.

In [81], Suayip Yuzbasi gave numerical treatment for high order linear singular DDEs

$$\sum_{\beta=0}^{m}\sum_{\gamma=0}^{n} P_{\beta\gamma}(l) y^{(\gamma)}(\alpha_{\beta\gamma} l + \delta_{\beta\gamma}) - l(l-c)f(l) = 0, \qquad 0 \leq l \leq c, \quad (54)$$

along with boundary conditions:

$$\sum_{\gamma=0}^{n-1}\left[a_{\zeta\gamma}y^{(\gamma)}(0)+c_{\zeta\gamma}y^{(\gamma)}(c)\right]=\lambda_\zeta \qquad ;\zeta=0,1,\ldots,n-1, \quad (55)$$

whereas $\alpha_{\beta\gamma}$, $\delta_{\beta\gamma}$, $a_{\zeta\gamma}$, $c_{\zeta\gamma}$ and λ_ζ are limited set of constants; $y^{(0)}(l) = y(l)$ is unidentified funtion, and $P_{\beta\gamma}(l)$, $f(l) \in C(0, c)$, $P_{\beta\gamma}(l)$, $f(l)$ may be undefinable at the points $l = 0$ and $l = c$. In this study, the author developed numerical method for eq. (54). Using collocation points and Bessel polynomials, he translated the singular DDEs in the matrix equation. If the exact solutions are given as polynomials, then analytical solutions are obtained by using this method. He established the validity and competence of the presented approach by presenting some examples numerically and comparing them with the results in literature. Numerical results are calculated by using Matlab (R2008a).

In [82], Gepreel, Shehata gave rational Jacobi elliptic function scheme to solve nonlinear DDEs

$$\frac{dy_n(r)}{dr}=(\mu+\nu y_n+\omega y_n^2)(y_{n-1}-y_{n+1}). \quad (56)$$

Here μ, ν, ω are non-zero constants. The authors produced rational Jacobi elliptic solution for nonlinear DDEs by the rational Jacobi elliptic scheme. By applying the lattice equation, the proposed method is capable of yielding a range of exact solutions for nonlinear DDEs in mathematical physics. These solutions encompass hyperbolic function solutions and trigonometric function solutions, particularly in cases where the modulus m tends toward both 1 and 0.

In [83], Salih Yalcinbas and Tugce Akkaya integro DDE

$$\sum_{m=0}^{s}P_m(x)y^m(x)+\sum_{n=0}^{t}P^*_{(n)}(x)Y^{(n)}(x-\tau)=g(x)+\int_a^b K(x,t)y(t)dt, \quad ;-\infty<a\le x,\quad t\le b\le\infty, \quad (57)$$

$$\sum_{m=0}^{s-1}[\alpha_{im}y^{(m)}(a)+\beta_{im}y^{(m)}(b)+\gamma_{im}y^{(m)}(\eta)]=\nu_i, i=0,1,...,s-1, \quad (58)$$

where the functions $P_m(x)$, $P^*(n)(x)$, $K(x, t)$, and $g(x)$ are known and defined within the interval $a \le x$, $t \le b$. The constants $\alpha im, \beta_{im}, \gamma_{im}, \nu_i$ can be either complex or real numbers. The function $y(x)$ is the unknown quantity. The authors introduced Boubaker polynomials based collocation approach to find solutions for Riccati DDEs subject to mixed conditions. They compared the results obtained with those available in the existing literature to demonstrate the efficiency of the method.

In [84], Yuzbasi gave a numerical treatment for DDE of Riccati type

$$P(r)g'(pr+\alpha)+Q(r)g(qr+\beta)+R(r)g^2(mr+\gamma)=S(r), 0\le a\le r\le b<\infty, \quad (59)$$

along the conditions

$$\mu y(a)+\nu y(b)=\lambda. \quad (60)$$

Here the function $g(r)$ is unknown, and the known functions are $P(r)$,$Q(r)$,$R(r)$,$S(r)$ in interval $a \le r \le b$; $p, q, \alpha, m, \beta, \gamma, \mu, \nu$ and λ are constants belong to complex or real numbers. The approximate solution for equation (59) with mixed conditions was achieved through a collocation approach relying on the Bessel functions of the 1st kind. Error analysis, as outlined in references [85] and [86], was employed for assessment. To affirm the effectiveness and reliability of their method, the authors presented several numerical examples and conducted comparisons with other methods from the existing literature. The numerical results were computed using Maple 9.

In [87], Gokmen and Sezer presented the Taylor collocation approach for solving the high order linear DDEs

$$\sum_{\beta=0}^{m}\sum_{\gamma=1}^{n}P_{\alpha\gamma}(r)y_\gamma^{(\beta)}(\mu r+\nu)=g_\alpha(r), \alpha=1,2,...,n, \quad (61)$$

along with mixed conditions as

$$\sum_{\alpha=0}^{m-1}[a^k_{\alpha\beta}y_k^{(\alpha)}(a)+b^k_{\alpha\beta}y_k^{(\alpha)}(b)+c^k_{\alpha\beta}y_k^{(\alpha)}(c)]=\nu_{\alpha\beta}. \quad (62)$$

In this context, the unknown functions are denoted as $y_\alpha(r)$, $P_{\alpha\gamma}(r)$, and $g_\alpha(r)$. The functions $P_{\alpha\gamma}(r)$ and $g_\alpha(r)$ are known and defined within the interval $a \le r \le b$. Additionally, there are constants $a_{\alpha\beta}$, $b_{\alpha\beta}$, $c_{\alpha\beta}$, and $\nu_{k\beta}$ that have been appropriately selected. The authors introduced a Taylor collocation approach, it is used to solve systems of higher order linear DDEs and relies on Taylor polynomials. This approach transformed systems of DDEs and associated conditions into matrix equations, incorporating unknown Taylor coefficients through the utilization of Taylor collocation points. The authors introduced a Taylor collocation approach based on Taylor polynomials for the solution of linear higher order DDEs. By applying this approach, they were able to derive a novel set of equations from the matrix equation, which corresponded to linear algebraic systems. To demonstrate the validity and effectiveness of their proposed method, they presented several numerical examples and compared their results with other techniques available in the existing literature.

In [88], Polyanin, Zhurov established exact solution of non-linear DDE of incompressible viscous fluid

$$\begin{aligned}[v_t+(v.\nabla)v]_{t+\tau}&=-\nabla\bar{p}+\nu\triangle v,\\ \nabla.v&=0.\end{aligned} \quad (63)$$

The authors applied a finite relaxation time approach to calculate exact solutions for nonlinear DDEs describing incompressible viscous fluid behavior. These results were then used to address several hydrodynamic problems. The fluid flow was characterized by longitudinal periodic oscillations around a rigid plane, considered as one-dimensional. In contrast, the flow near the attached porous plate, which involved a pressure gradient, was analyzed as two-dimensional. In addition, the investigation examined problems pertaining to the hydrodynamic volatility of these solutions and methods for improving the fluid DDEs and the corresponding heat and diffusion equations. The presented modified relaxation fluid model holds the potential to explain the onset of turbulence under certain circumstances.

In [89], Aslan consider the fractional type DDE

$$\dot{v}_n=R(...,v_{n-1},v_n,v_{n+1},...). \quad (64)$$

Here $vn(t) = v(n,t)$; $n \in Z$ the nth particle's position of the equilibrium position. ChatGPT The author employed an extended simplest equation scheme to solve fractional-type DDEs. Applying a real discrete Miura transforming, they studied the discrete KdV formulation and systems associated with well-known self-dual network difficulties. Utilizing symbolic calculations, the author derived various kinds of exact solutions, including trigonometric, rational, and hyperbolic solutions. The development of numerical systems may benefit greatly from these discoveries as a foundation. In [90], Stevic, Diblik and Smarda consider systems of DDEs

$$\acute{x}(r) = Ax(r) + Bx(r-1) + C\acute{r}(r-1) + f(r, x(r), x(r-1), \acute{x}(r-1)). \quad (65)$$

Here the variable *r* belongs to the real numbers, and the matrices *A*, *B*, and *C* are all real ($n \times n$) matrices. The function f is defined over the real numbers and maps to $R^n \times (R^n)^3 \rightarrow R$. For equation (65), the authors developed adequate criteria for determining the presence of a periodic C^1 solution. They also explored the presence of a parametric family of solutions, characterized by '*n*' parameters, which approach zero along with their first derivatives.

In [91], Aslan consider the time fractional DDE

$$D_r^\beta y_n = R(y_{n-1}, y_n, y_{n+1}), 0 < \beta \leq 1. \quad (66)$$

The author has introduced an analytical method for solving equation (66) using Jumarie's modified Riemann-Liouville derivative with a time order of β. In this equation, the dependent variable y_n is a function of $y(n, r)$, where n belongs to the set of integers *Z*, representing a lattice variable. The study employed this approach to explore time-fractional DDEs with a rational description using symbolical computation. He demonstrated a rational, trigonometric, and hyperbolic type of exact solutions by giving three different examples of eq. (66) and its system.

In [92], Wen and Wang consider the non-linear Schrodinger DDE

$$ip_r + p_{yy} \pm 2p|p|^2 = 0. \quad (67)$$

They formulated conservation laws and Darboux transformation with N folds for the non-linear Schrodinger DDE depends on its Lax pair. The obtained Darboux transformation is used for deriving odd soliton solutions in the form of a determinant. The inelastic interaction phenomena between '3' solitons for the eq. (67) are graphically demonstrated.

In [93], Guo et. al consider the DDE

$$\dot{v}(r) + B(r)v(r) + f(r, v(r+\tau), v(r), v(r-\tau)) = 0. \quad (68)$$

The authors established the homoclinic solutions to the DDE. The authors introduced a variational framework for eq. (68), incorporating periodic boundary value conditions. The associating theorem for the existence of homoclinic solution is considered. The supporting Ambrosetti Rabinowitz growth condition is proved for the 2nd existence result of eq. (68).

In [94], Jun Shena and Wei Xing Zheng consider the linear coupled DDE along with time fluctuating delays

$$\dot{x}(l) = Ex(l) + Fy(l - \tau(l)), \quad (69)$$

$$y(l) = Gx(l) + Hy(l - \tau(l)). \quad (70)$$

here $x(l) \in R^m$, $y(l) \in R^n$. The time fluctuating delay $\tau(l)$ is supposed to be bounded and continuous i.e., $0 \leq \tau(l) \leq \tau$ τ some +ve constant. The initial condition regarding system (69) is presented by $x(0) = \phi$, $y(l) = \psi(l)$; ($l \in [-\tau, 0)$) and $\phi \in \mathbb{R}^m$ and $\phi \in PC([-\tau, 0), \mathbb{R}^n)$.

The asymptotic stability of a particular class of eq. (69) among positive internal property is analyzed. Explicit characterization of the positivity of eq. (69) is discussed. The asymptotic property of their case trajectories originating from suitably taken initial conditions and entrywise monotonicity depend on the positivity (69) including constant delays are examined. Moreover, the comparison made between the time varying system with delay and analogous constant delay system. It is observed that the asymptotical stability of an internally +ve coupled DDEs relies on the delay free system.

In [95], Geng et. al gave solution of singularly perturbed DDE numerically with delays

$$\delta v''(t) + a(t)v'(t-\eta) + b(t)v(t) = g(t) \qquad \text{for } 0 < t < 1, \quad (71)$$

$$v(t) = \psi(t) \quad \text{for } t \in [-\eta, 0], \quad v(1) = \alpha. \quad (72)$$

This code separates the equations and aligns them using the align environment while maintaining proper spacing and alignment. here $0 < \delta \ll 1$ and η is a parameter with small delay, $\eta = O(\delta)$, $a(t)$, $b(t)$, $g(t)$ are smooth functions. They demonstrated the numerical solution that displaying the behavior of boundary layer for eq. (71). For the solution of eq. (71) the reproducing kernel scheme shown in the already existing literature is not adequate. The authors refined the reproducing kernel method to get an authentic estimation of considered equation under discussion.

In [96], Prakash et. al consider the nonlinear fractional DDE

$$D_r^\beta[z_n(r)] + R[x, r, z_n, z_{n-i}, z_{n+i}] + N[z_n, z_{n-i}, z_{n+i}] = 0, i \in N, 0 < \beta \leq 1, \quad (73)$$

along with initial conditions

$$z_n(0) = g(n), \quad (74)$$

here he Caputo fractional derivative is $D_r^\beta z_n$ (r) of the function and R represent the remainder term, $N[z_n, z_{n-i}, z_{n+i}]$ is the nonlinear term. The authors presented a modified He Laplace approach for solving eq. (73) with space and time variable. The authors presented a modified He Laplace approach for solving eq. (73) with space and time variable. Laplace transforms method and the fractional- homotopy perturbation approach is combined, and modified He Laplace method is developed. He's polynomials could be used to transform the nonlinear terms. The proposed scheme had been successfully applied to the discrete modified KdV and modified Lotka Volterra equations. A rapidly convergent series is obtained by employing the present scheme.

In [97], Balci and Sezer consider the linear Fredholm integro DDEs having constant arguments as well as variable coefficients

$$\sum_{i=0}^{n_1}\sum_{k=0}^{l_1} P_{ik}(r)y^{(i)}(r+\tau_{ik}) = \sum_{s=0}^{n_2}\sum_{q=0}^{l_2}\int_0^c K_{lt}(r,l)y^{(s)}(l+\alpha_{ik})dt + f(x), \quad (75)$$

along with mixed conditions as

$$\sum_{i=0}^{n_1-1}(d_{ij}y^{(i)}(0) + c_{ij}y^{(}i))(c)) = \delta_j, j = 0, 1, ..., n_1 - 1, n_1 \geq n_2. \quad (76)$$

Here, the known functions are $P_{ik}(r)$, $K_{lt}(r, l)$, and $f(x)$, all defined within the interval $0 \leq r,l \leq b < \infty$. Additionally, there are various constants denoted as τ_{ik}, α_{ik}, δ_j, d_{ij}, and c_{ij}. The objective is to find the unknown function $y(r)$. The authors have presented a numerical approach for solving equation (76) while considering initial boundary conditions. To solve this equation, they employed a decomposition method that transforms it into a system of algebraic equations. This transformation was achieved by using Euler polynomials and selecting appro- priate collocation points.

Moreover, the authors performed an error analysis with respect to the residual function pertaining to the solution technique.

In [98], Kurkcu consider integro DDE

$$\sum_{i=0}^{n_1}\sum_{k=0}^{K} P_{ik}(r)z^{(i)}(\lambda r+\nu) = f(x) + \int_a^b \sum_{l=0}^{n_2} K_l(r,t)z^{(l)}dt + \int_a^{\phi(r)} \sum_{m=0}^{n_3} K_m(r,t)z^{(m)}dt, \quad (77)$$

$$\sum_{i=0}^{n_1-1} (b_{ik}z^{(i)}(b) + c_{ik}z^{(i)}(c)) = \eta_i, i = 0,1,...,n_1-1. \quad (78)$$

Here the known function are $P_{ik}(r)$, $f(x)$, $K_l(r, t)$, $\phi(r)$, $K_m(r, t)$ defined in the interval $-\infty < b \leq r$, $t \leq c < \infty$; b_{ik}, c_{ik} , η_i are appropriate constants, and the function $z(r)$ is unknown. They presented a numerical treatment for linear integro DDEs having variable coefficients along with the mixed conditions. An error analysis approach is employed in the study to evaluate the validity of the numerical approach with a focus on the residual functions.

In [99], Ye et. al consider generalized form of DDEs

$$\ddot{v}_n = f_n(r, v_{n-1}, v_n, v_{n+1}, \dot{v}_{n-1}, \dot{v}_{n+1}). \quad (79)$$

They presented a symmetry classification algorithm based on enabled transformations, fundamental Lie point symmetries and Lie algebraic compositions. A Toda type lattice equation confirmed the efficiency of the proposed technique.

In [100], Yue Wang established the existence of meromorphic solutions and elucidated their properties by employing Nevanlinna theory, which deals with the value distribution of meromorphic functions. The outcomes of this investigation were found to be more precise compared to the previously available results in the literature. To illustrate the accuracy of their findings, the author provided specific examples.

In [101], Lingyun Gao consider the complex system of DDEs

$$W_1'(z)^{l_1} + W_2(z+d)^{n_1} = G_1(z), \quad (80)$$
$$W_2'(z)^{l_2} + W_1(z+d)^{n_2} = G_2(z), \quad (81)$$
$$W_1'(z)^2 + W_2^2(z+d)^{n_1} = G_1(z), \quad (82)$$
$$W_2'(z)^2 + W_1^2(z+d)^{n_2} = G_2(z). \quad (83)$$

The author discussed the system of complex DDEs of two types with finite order and obtained the entire solution. He extended some of the results of complex DDE to the system of complex system of DDEs.

A new method based on the replicating kernel Hilbert space approach and the Gram-Schmidt orthogonalization procedure was presented by Sahihi et al. [102]. This approach is designed to address singularly perturbed DDE exhibiting boundary layer behavior. In particular, the equations require a negative change in the term for the derivative for neighbourhood points denoted by parameter r, which can be either 0 or 1.

$$\epsilon y''(l) + p(l)y'(l-\eta) + q(l)y(l) = g(l) \qquad \text{for } l \in [0,1], \quad (84)$$

$$y(l) = \psi(l) \quad \text{for } l \in [-\eta, 0], \quad y(1) = \beta, \quad (85)$$

here ϵ, η are small arguments, and $0 < \epsilon \ll 1, 0 < \eta \ll 1$. The smooth functions are $p(l)$, $q(l)$, $g(l)$, $\psi(l)$ and β is fixed, $q(l) \leq -\vartheta < 0$, $p(l) \geq N > 0$ and ϑ, N are positive generic constants. The domain of the presented problem was split into two subintervals by the authors: one had a boundary layer, while the other did not. Their proposed method yielded meaningful results, particularly when the boundary layer is situated on the left side. In the case of equation (84), it was necessary to adjust the variable to account for changes within the boundary layer region.

In [103], Pathirana et. al consider the coupled DDEs having time varying delays

$$\dot{u}(r) = Cu(r) + Dv(r-\tau(r)), r \geq 0, \quad (86)$$
$$v(r) = Eu(r) + Fv(r-w(r)), \quad (87)$$

here the state vectors are $u(.) \in \mathbb{R}^m$, $y(.) \in \mathbb{R}^n$. In the context of the problem, several constant matrices are known, including $C \in \mathbb{R}^{m\times m}$, $D \in \mathbb{R}^{m\times n}$, $E \in \mathbb{R}^{n\times m}$, and $F \in \mathbb{R}^{n\times n}$ Notably, F is supposed to be a Schur matrix. The variable in time delays are represented by $\tau(t) \in \mathbb{R}\geq 0$ and $w(t) \in \mathbb{R}\geq 0$. The authors addressed the stability problem arising from a particular class of DDEs that are positively associated and contain unbounded variable in time delays. This technique is based on applying a decreasing function to determine the upper bounds for the state vector. This approach was employed to assess and verify the stability of the system. This method did not practice the comparison approach, or the usual Lyapunov Krasovskii functional approach depends on positive systems having constant delays as in existing techniques. New asymptotic stability of the system having boundless variable in time delays is derived. A numerical example using simulation is provided to validate the stability requirement.

In [104], Sirisha et. al introduced a mixed finite difference approach for solving singularly perturbed DDEs

$$\epsilon u''(r) + \alpha(r)u'(r) + \beta(r)(r-\eta) + \gamma(r)u(r) + \omega(r)u(r+\delta) = g(r), 0 < r < 1, \quad (88)$$

under the conditions

$$u(r) = \psi(r), -\eta \leq r \leq 0, \quad (89)$$
$$u(r) = \phi(r), 1 \leq r \leq 1+\delta, \quad (90)$$

here $0 < \epsilon \ll 1$ is perturbation parameter, η is the delay and δ is the advance parameter $\alpha(r)$, $\beta(r)$, $\gamma(r)$, $\omega(r)$, $g(r)$, $\psi(r)$, $\phi(r)$ are continous differentiable and bounded functions. The authors used mixed finite difference technique for solving eq. (88). The authors adopted a decomposition approach to address the problem at hand, leading to solutions exhibiting boundary layer behavior primarily in the direction of the interval's left end. By adding a terminal point inside the domain, they divided the problem in two different parts: the inner and outer regions. This transformation effectively reduced the given problem into an equivalent asymptotically singular perturbation problem. They then treated the problem separately for the inner and outer regions, employing the mixed finite difference technique for both. Various choices of the terminal point were utilized in applying the proposed method. To assess the reliability and effectiveness of their approach, the authors conducted convergence and capability tests using different illustrative examples.

In [105], Sunil Kumar et. al consider the singularly perturb DDE of elliptic type

$$-\epsilon\psi''(r) + c\psi'(r-\nu) = q, \forall r \in \Omega, \quad (91)$$
$$\psi(r) = p(r) \geq 0, -\mu \leq r < 0, \psi(1) = p(1) \geq 0. \quad (92)$$

The authors presented a complete flux scheme depends on the finite volume for the solution of eq. (91). They developed alternate integral descriptions for the flux that performs a vital role in the complete flux scheme's derivation. They proved the consistency, stability, and quadrature convergence for the proposed approach.

Numerical examples are successfully solved by applying the proposed approach.

In [106], Rong WuLu consider the integrable differential difference equation

$$\mu_{n,t} = \mu_n \nu_n (\mu_n - \mu_{n-1}), \tag{93}$$

$$\nu_{n,t} = \mu_n \nu_n (\nu_{n+1} - \nu_n). \tag{94}$$

The authors devised a one-fold Darboux Backlund transformation for equation (93), building upon the approach detailed in reference [107]. They accomplished this by employing a suitable gauge transformation matrix, akin to a Lax pair framework. Subsequently, the one fold Darboux Backlund transformation was extended to an N fold Darboux Backlund transformation by employing it N times. This N-fold transformation was then applied to derive two exact solutions for the equation.

In [108], Dong and Liao gave meromorphic DDE's solutions in general of the form

$$L(z,g) + \sum_{i=1}^{p} a_i(z) g^{ki}(z+\eta_i) = \sum_{i=1}^{l} \alpha_i e^{\lambda_i z}, \tag{95}$$

here $L(z,g) \not\equiv 0$ represents linear differential difference polynomial of g with small coefficient function. Non-vanishing constants are $1 < k_1 < ... < k_p, \alpha_i (1 \leq i \leq q$, and non-vanishing small functions are $a_i(z)(1 \leq i \leq p$. In addition, the symbol $S(l, f)$ indicates the small quantum of function $g(z)$ fulfilling $S(l,f) = o(T(l,f))$ as $l \to \infty$ outside of a feasible remarkable set with finite logarithmic measurement. They employed Nevanlinna theory [109,110]. They also consider the symbols $\sigma(g) = limsup_{l\to\infty} \frac{log^+ T(l,g)}{logl}$, $\lambda(g) = limsup_{l\to\infty} \frac{log^+ N(l,\frac{1}{g})}{logl}$. If the inequalities $\lambda(g) < \sigma(g), \lambda(1/g < \sigma(g)$ are satisfied, then equation (95) yields transcendental meromorphic solution g of finite order. Then the relationships among coefficients in $\phi(z,g)$ and α_i, λ_i and representation of g can be achieved.

In [111], Sahihi et. al consider singularly perturbed DDEs having delay

$$\epsilon y''(l) + p(l) y'(l-\eta) + q(l) y(l) = g(l), \quad l \in [0,1], \tag{96}$$

$$y(l) = \psi(l), l \in [-\eta, 0], y(1) = \beta, \tag{97}$$

here ϵ, η are small arguments, and $0 < \epsilon \ll 1, 0 < \eta \ll 1$. The smooth functions are $p(l)$, $q(l)$, $g(l)$, $\psi(l)$ and β is fixed, $q(l) \leq -\vartheta < 0$, $p(l) \geq N > 0$ and ϑ, N are positive generic constants. The authors used the reproducing kernel Hilbert space technique depend on the collocation approach for the solution of eq. (96). The authors implemented a technique that involves the Gram-Schmidt orthogonalization process to address equation (96). Both boundary layer and small-delay oscillatory behaviour are shown by this equation. There are two subintervals in the domain of the problem: one with a boundary layer and the other without. It is shown form given numerical examples that the proposed method gives rapid convergence with compact computational labor.

A novel approach is introduced, involving an exponentially adapted three-term finite difference method, to numerically approximate a boundary-value problem associated with a singularly perturbed 2nd order DDE. This DDE features both negative delayed and positive advanced shifts, and the study by Ranjan et al. [112] addresses this problem. The approach leverages Taylor's series expansion to create an approximation of the problem, and subsequently, employs finite difference approximation techniques to establish a new three-term recurrence relationship. Additionally, a new exponential fitting factor is incorporated to the obtained technique employing the notion of singular perturbations, and the resultant tridiagonal system of equations is solved using an effective "discrete invariant imbedding algorithm."

Anwar et al. [113] introduced a stochastic paradigm driven by artificial intelligence capabilities to numerically address nonlinear DDEs. They utilized this approach to study the dynamics of plant virus propagation while considering the effects of seasonality and delays. Their method involved the implementation of neural networks combined with a Bayesian regularization trchnique for improved accuracy and robustness in solving the DDEs systems.

Shoaib et al. [114] explored the dynamics of a non-linear SEIR model with multiple delay terms, focusing on worm transmission in wireless sensor networks. They employed an intelligent numerical computing approach, harnessing neural networks and the Bayesian regularization approach. The mathematical model under investigation pertained to a system of DDEs that characterized the dynamics of wireless sensor networks.

3. CONCLUSION

In this study, comprehensive literature concerning research assignments for solving DDEs using analytical and numerical approaches has been presented. Initially, researchers found the presence, uniqueness, and stability of periodic solutions of DDEs. Some researchers discussed the asymptotic behavior of solutions of DDEs. As all the DDEs are difficult to handle analytically. so most of the researchers keep focusing to tackle such problems numerically.

New theorems with their proofs has been presented successfully by using differential transform approach to the solution of DDE. DDEs solved by applying Taylor polynomial approach by calculating the coefficients using Taylor expansion. The truncation limit of N would be chosen largely to get the best approximation. A new algebraic algorithm to develop traveling wave solution has been presented. Nonlinear differential difference problems have not been handled directly by the homotopy analysis method earlier. But the authors have found the solution to such problems by using homotopy analysis method successfully. A numerical treatment to high order linear singular differential difference problems has been obtained by employing the first kind Bessel polynomials. Several rational Jacobi elliptic solutions to non-linear DDEs by the lattice equation have been presented, while the modulus ($m \to 1$, $m \to 0$), trigonometric function solutions, and hyperbolic function solutions have been acquired. An analytical solution of the equation is obtained from the exact solution which is polynomial, an intriguing characteristic of this technique. This method can be easily used to find the approximate solution via computer coding composed in matlab R2008a. A nonlinear DDE of incompressible viscous fluid including a finite relaxation rate has been presented. The model consists of just one new rheological parameter, τ is the relaxation time. Several exact solutions to such models have been derived. Conservation laws and N fold Darboux transformation forthe non-linear Schrodinger DDE depends on its Lax pair have been constructed. As a result, odd soliton solutions have been derived based on the determinant. Homoclinic solutions to the DDE have been presented. The associating theorem for the existence of a homoclinic solution is analyzed. A symmetry classfication algorithm based on enabled transformations, fundamental Lie point symmetries, and Lie algebraic compositions have been presented for generalized DDEs. High dimensional Lie-algebras are required to develop symmetry classification. The classification regarding Liealgebras turns fiercely more complex by the increase of dimension and needs to accord with an extensive quantity of isomorphism classes. Meromorphic solutions of DDEs have been presented.

The stability analysis in this study relies on linear matrix inequalities to solve coupled DDEs with multiple delays, especially in the context of channels. The approach significantly reduces the problem's size when compared to conventional formulations that deal with differential-difference problems. Additionally, the authors introduced the Taylor collocation scheme as a method for solving high-order linear systems of DDEs. It was observed that the solutions obtained using this scheme are more reliable than those achieved through methods found in existing literature. The study also established sufficient conditions for the existence of periodic C^1 solutions within the system of DDEs. Furthermore, necessary and sufficient conditions for asymptotic stability were provided for coupled delay DDEs involving time-fluctuating delays while preserving internal-positive characteristics. Various properties related to the asymptotic behavior and monotonicity of such problems, as well as initial conditions, were identified. The research presented the analysis of asymptotic stability for positively coupled DDEs with unbounded time variable delays.

The study successfully derived an integral representation for the classical solution to the Cauchy problem of a parabolic DDE. Furthermore, generalized conditional symmetries were employed as an effective method for solving problems that exhibit a limited number of Lie point and conditional symmetries. These conditional symmetries were utilized to discover new exact solutions for DDEs. To explore the Lie symmetries associated with DDEs, the work of Estabrook and Harrison was extended. In addition, the discrete exterior differential method was applied to analyze the symmetry properties of the (2+1) dimensional Toda problem within this context. The study also presented traveling wave solutions for nonlinear polynomial DDEs through explicit integra-tion methods. A straightforward algorithm known as the symmetrical Fibonacci tane function approach was introduced in this context. $\left(\frac{G'}{G}\right)$ expansion method has been presented to solve nonlinear DDE, trigonometric-function solutions and hyperbolic-function solutions along with parameters have been obtained. Special solutions, including singular traveling wave solutions and kink-type solitary wave solutions, were obtained by setting specific parameters to special values. Exact solutions for nonlinear DDEs were derived using the exponential function-rational expansion approach. Additionally, the Adomian decomposition technique was utilized to solve a nonlinear DDE, leading to exact solutions for discrete mKdV problems and the Volterra problem. The soliton solution of the discrete mKdV problem by using Pade approximation has been obtained that quite matches the exact solution. Entire solutions to non-linear difference as well as partial DDEs of the Fermat type have been presented.

The study showcased the existence of meromorphic solutions and explored their properties within the realm of complex DDEs, employing Nevanlinna theory to analyze the value distribution of meromorphic functions. Notably, the results obtained were more accurate compared to the existing findings in the literature. Moreover, the research successfully derived entire solutions for two types of complex DDEs with finite orders. Several results pertaining to complex DDEs were extended to complex DDE systems, expanding the scope of the study.

A numerical approach for solving Riccati DDEs was introduced, utilizing a collocation scheme based on first-kind Bessel functions. However, it's important to note that the solution might not be valid when the parameter "N" is large due to issues related to polynomial interpolation. When using a computer algebraic system, computational errors can become significant for large "N." The study also calculated trigonometric, hyperbolic, and rational types of exact solutions for fractional DDEs by choosing appropriate parameter values. This method is effective, accurate, and easily implementable using computer software such as Mathematica. Time fractional DDEs with a rational type were modeled using an enhanced version of the $\left(\frac{G'}{G}\right)$ expansion technique. Symbolic computation systems like matlab, Maple, and Mathematica played a crucial role in these calculations. Fractional complex transmutations were employed to convert fractional differential equations, including Jumarie's discretion, into ordinary differential equations. These transmutations were particularly applicable to wave solutions, although not all forms of fractional order differential-difference equations could be accommodated by this method. The study also introduced the Modified He Laplace method for solving nonlinear fractional differential-difference problems. This method provides solutions in the form of nonlinear fractional DDEs, which can be directly measured without requiring perturbation, linearization, or conditional assumptions. Furthermore, the infinite series obtained in this method exhibits rapid convergence to the exact solution.

The study focused on the numerical solution of singularly perturbed DDEs that exhibit rapid oscillations with both positive and negative shifts. The amplitude of these oscillations is influenced by the shifts, either decreasing or increasing, depending on whether the negative shift dominates or the positive shift dominates, respectively. For both types of shifts, the period of oscillations remains unchanged as long as the shifts are of the order of 'ϵ^2'. Specifically, a second-order singularly perturbed DDE, featuring a negative shift in its first derivative, was solved using the B-spline collocation method in combination with a fitted mesh. This method achieved approximately second-order parameter uniform convergence. The research also tackled the numerical solution of time-dependent singularly perturbed DDEs involving advanced and delay terms with a boundary layer. The technique employed B-spline collocation for the spatial direction, employing a piecewise uniform mesh, and an implicit Euler technique for transient discretization of ordinary differential equations in the time dimension. Importantly, this technique demonstrated uniform convergence concerning both the perturbation parameter and the mesh parameter. Furthermore, singularly perturbed DDEs exhibiting boundary-layer behavior were addressed using an improved reproducing kernel approach, specifically within the framework of the reproducing kernel Hilbert space method. The problem was divided into two subintervals, and the approach yielded a good approximation by selecting distinct values of δ and ϵ. However, this method was particularly effective when the boundary layer occurred within the subinterval containing significant singularity. It may not be as suitable when dealing with points having exponential properties throughout the entire [0, 1] interval. Lastly, singularly perturbed DDEs with mixed shifts were solved using domain-decomposition techniques. The obtained solutions exhibited boundary-layer behavior at the left end of the interval. These domain-decomposition techniques originated from Prandtl and are well-suited for addressing singular perturbation problems. The solution's behavior primarily at the boundary layer could be illustrated by using such techniques. Also, the Numerov technique with the mixed-finite differences and Non-symmetric finite differences have been utilized to handle the 1st derivative. This method has been used for solving inner as well as the outer region. The final point x_1 has shown iterative behavior, the procedure is repeated to several values of x_1 as for as the solution is sustained in inner as well as the outer region. Elliptic singularly-perturbed DDEs have been tackled using a complete flux technique.

Reference function based element-wise inhomogeneous boundary value problems has been used to obtain the fluxes with specific 2 elements originating from the particular and homogeneous solutions of the boundary value problems. Numerical fluxes have been derived by using inhomogeneous fluxes composed owing to Green's function in addition to an appropriate selection of quadrature rules.

Linear integro DDEs have been solved numerically by employing the Chebyshev collocation method. The Chebyshev polynomial coefficients can be easily obtained by utilizing the computer programs is a remarkable benefit of this method.The shifted Legendre tau technique has proven to be an effective approach for addressing higher order linear Fredholm integro DDEs that involve variable coefficients. The study provided a numerical solution to these linear integro-differential-difference problems using the Boubaker collocation scheme. A notable feature of this method is its ability to extend the associated functions in the problem into the Boubaker series. Moreover, determining the coefficients of the Boubaker polynomial efficiently through computer code is another significant advantage of this scheme. Fredholm integro-DDEs were also addressed using a collocation method based on the Euler Taylor polynomial. This approach offers two primary benefits: it aids in the development of matrix equations and is computationally efficient, resulting in reduced computing time. In addition, the numerical solution for linear integro DDEs was obtained by employing Dickson polynomials. The main advantage of this technique lies in its ability to extend the functions in the equation into Dickson polynomials. To achieve the most desirable approximation, it is essential to choose a sufficiently large truncation limit, denoted as 'N '. Furthermore, the study derived a one-fold Darboux-Backlund transformation by employing an appropriate gauge transformation matrix similar to a Lax pair for an integrable system of DDEs. Subsequently, the N-fold Darboux-Backlund transformation was derived by utilizing this one-fold transformation multiple times. Artificial intelligence techniques were used to analyze the dynamics of plant virus and wireless sensor network systems, providing a novel approach to numerically treat systems with delay differential equations.

In the future, one may used the capabilities of computing paradigms centered around artificial neural networks and their deep variants to numerically address differential and difference models [115–120].

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