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Numerical Analysis on Nonlinear Chaotic Financial System: Comparative scrutiny of Solver Performance on System Dynamics

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ABSTRACT

Financial system dynamics are inherently complex, nonlinear and uncertain often characterized by the chaotic, random behavior and interactions. Understanding the complex financial-based dynamics within the society is crucial for designing effective economic policies to achieve sustainable economic goals, raise living standards and prosperity among Nations. This study addresses the nonlinear chaotic financial system (NCFS) model governed by the ordinary differential equations (ODEs), incorporates the three compartments related to loan interest rate, demand for capital investment and market inflation index. The investigation offers a comprehensive analysis of the system behavior by varying the NCFS model parameters and initial conditions of the state variables resulting into various cases and variants. The formulation of different case studies allows to examine system sensitivity as how the NCFS responds to changes in saving propensity, expenditure on investment, and sensitivity to price changes (market demand responsiveness). The complex NCFS dynamics are analyzed with the employment of four numerical solvers including Livermore solver, backward differentiation formula (BDF), explicit Runge-Kutta and implicit Runge-Kutta methods. The exploited numerical solvers prove to simulate accurate and efficient NCFS dynamics depicted by the comparative analysis among each numerical method. The efficiency of numerical solvers in generating the real-worl data system behavior for NCFS is evaluated by absolute error analysis. The detailed error analysis provides the insights that the error is minimum and close to zero for all the formulated NCFS cases and associated variants. The comprehensive analysis opens the path for practical implications in robust nonlinear modeling, numerical simulations and predicting complex financial systems.

Keywords: Backward Differentiation Formula; Explicit Runge-Kutta; Nonlinear dynamic systems; Financial Econometrics; Implicit Runge-Kutta; Livermore Solver.

1. INTRODUCTION

The economic and financial-based systems dynamics are inherently stiff, nonlinear octenyl characterized by complex, uncertain interactions within the social system that results in chaotic and probabilistic behavior [1-2]. These random, complex and ambiguous dynamics are crucial to understand for designing effective policies to combat the challenges associated to the economic instability and crises [3-5]. However, modeling market fluctuations [6-7], forecasting market failures [8-9], energy insecurity [10-12], prediction of economic crises [13-14] before the event occurs, requires the innovative strategy for recommending and implementing the effective policies to attain economic sustainability for longer periods of time [15-16]. Mostly the research has been seen to exploit traditional methodologies for financial system analysis that are largely relied on real-world, empirical data as well as linear approximations [17-20]. Though these conventional methodologies are often unable to capture the intricacies, complexities of chaotic behaviors and stiff, nonlinear dynamics observed in the real-world scenarios [21-23]. Researchers are now interested to address this matter by giving special attention to nonlinear financial-econometric systems that are governed with ordinary differential equations (ODE) system that offers the comprehensive analysis of the underlying dynamics of system behavior in response to changing parameter values and initial conditions [24 -25].

Mathematical modeling provides a powerful framework particularly for analyzing complex market and social dynamics [26-27]. By providing a mechanism through the exploitation of ODEs the dynamic, chaotic and nonlinear interactions within the socio-economic system can be captured easily and accurately [28]. The models offer the researchers to efficiently represents the state variables such as investment demand, inflation index [2], financial crime committed individuals, asset price [29-30], foreign direct investment, unemployed individuals and corrupt individuals [31-32] along with the interdependencies in the form of model parameters like demand responsiveness, saving rates, investment expenditures, workforce retirement rate, and crime rates. Exploitation of ODEs can assist policy makers, data analysts and field experts to

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design more effective policy interventions by simulating how the changes in market conditions, economic shifts or policies propagated through the real-world system [33-35]. In the context of social sciences, mathematical modeling has the ability to bridge theoretic understanding and practical implications by creating linkages between the individual conduct to macroeconomic indicators [26, 36-37]. This allows for the exploitation and exploration of various real-life scenarios with varying conditions such as changes consumer preferences, market conditions, price fluctuations and other social-economic phenomena while providing the predictive capabilities that goes beyond static evaluation [38-40].

The issues related to the sparse or incomplete real-world data can also be resolved with the employment of numerical approaches that enables a critical feature of hypothesis testing and comparison of various numerical methods, as most of the time real-world data is unavailable and the data source is not authentic leading to biased datasets [41-42]. The simulation of the dynamic systems facilitated by the numerical approaches enhances the machine learning techniques capability from a perspective of data analysis crucial for determining feedback loops, tipping points and critical thresholds that influence financial stability and chaotic behavior. In addition, predictive accuracy of the model can be increased that supports robust data-driven insights, decision-making and effective policy implications under uncertain environments [43-44]. The numerical approaches like the backward differentiation formula (BDF), explicit Runge-Kutta (RKE), implicit Runge-Kutta (RKI) and Livermore solver approximate ODEs solutions to facilitate researchers in exploring complex and dynamic interactions in diversified fields of finance, social sciences, business, economics, engineering and physics [45-46]. These computational methodologies provide a framework for dealing with systems that differs in terms of complexity, nonlinearity and stiffness.

Each of the numerical solvers are capable of diverse strengths like BDF is a multistep method that utilizes previously computed input to approximate derivative of a function therefore, useful for solving stiff ODEs [47-48]. Runge-Kutta methods are the type of explicit and implicit iterative methods, generalization of Euler's method used to approximate solutions ODEs initial value problems and offers high accuracy. These methods can also be generalized for approximating solutions for stochastic differential equation (SDG) system [49-50]. Livermore solver provides stable and efficient approach for solving ODEs initial value problems, Livermore solver for ODE or LSODE can solve stiff, no stiff and linear systems, allows for method switching and uses dynamic storage allocation [51-52]. The present study utilizes the aptitudes of the numerical solvers to solve the nonlinear chaotic financial system (NCFS) model governed by the ODEs, comprising of loan interest rate, demand for capital investment and market inflation index as the three model compartments. The NCFS model parameters along with initial condition values of the key variables are altered to formulate various cases and association variations for robust analysis. For the sake of comparing the performance of different numerical techniques, this study employs the four numerical solvers i.e., BDF, RKE, RKI, and LSODE methods to generate the reference outcomes of the NCFS model that simulate real-world dynamics of the underlying financial system. A systematic framework for error analysis is also provided by comparing the performance of each numerical technique and evaluated with the aid of absolute error (AE) metric. Although this comprehensive approach contributes to novel insights into nonlinear social-economic and financial dynamics, also having theoretical and practical implications in various fields [53-56].

Objectives of the presented strategy based on NCFS model:

- To represent the dynamics of NCFS model using ODEs comprising the key variables of loan interest rate P(t), demand for capital investment Q(t) and market inflation index R(t).
- To explore the changes in NCFS model parameters i.e., saving propensity, expenditure on investment, and sensitivity to price changes and initial conditions influence the behavior of the underlying system.
- To formulate the various cases and associated variations by altering the values of NCFS parameters and state variables' initial conditions for robust analysis.
- To generate accurate synthetic data through the exploitation of various numerical solvers i.e., BDF, LSODE, RKI, and RKE.
- To compare the efficiency in the performance of various numerical solvers employed for solving the variants of NCFS model.
- To carry out the reliability assessment of the numerical solvers using the AE metric.

This research analysis is being segmented into the following five segments, where the first segment is dedicated for the introduction relevant to financial systems and related literature. Second segment discusses the mathematical formulation and necessary definitions of the NCFS model parameters and state variables. In segment 3 methodology of the proposed strategy is explained in detail for clear and better understanding whereas segment 4 carries out the comprehensive and detailed analysis of the results with discussion. Finally, the last segment 5 concludes the proposed strategy of the research work.

2. THE MATHEMATICAL FORMULATION OF STIFF NONLINEAR CHAOTIC FINANCIAL SYSTEM

The presented study takes into account the mathematical framework mentioned in [2] where the dynamics of the financial system is portrayed with the utilization of ODEs for NCFS model represented in Equations. 1, 2, and 3. The equations (1-3) presents the interactions among the state variables that are loan interest rate i.e., P(t), demand for capital investment i.e., Q(t) and market inflation index i.e., R(t) along with the incorporation of the NCFS parameters in the form of saving propensity, expenditure on investment, and sensitivity to price changes and initial conditions influence the behavior of the underlying system. The NCFS model in equations 1, 2, and 3 captures the nonlinear dynamics and chaotic behavior under certain conditions. The financial system proposed in [2] is analyzed and revisited to further understand the dynamics by varying the initial conditions of the key variables P(t), Q(t), and, R(t) and the NCFS parameter values. This variation allows for the formulation of four different cases and each case there has been associated 4 variations.

$$P'(t) = R(t) + (Q(t) - \eta)P(t), \tag{1}$$

$$Q'(t) = 1 - \lambda Q(t) - P^2(t), \tag{2}$$

$$\begin{aligned} &R'(t) = -P(t) - \mu R(t), \\ &Initial\,conditions: P(0) = p, Q(0) = q, R(0) = r \end{aligned} \tag{3}$$

The comparative analysis is offered with the employment of diverse numerical solvers that includes LSODE, RKI, BDF and RKE methods to solve the cases and variations of NCFS model. The necessary information regarding the model stability, formulation, derivation and sensitivity analysis is obtainable in the research work by [2]. The NCFS model necessary definitions, symbolic representations related to the parameters and the key variables are mentioned in Table 1 for a better understanding and clarity.

Table 1 The NCFS model state variables and parameters necessary definitions

Symbols	Definitions of the NCFS model state variables and parameters
$P(t)$	Loan interest rate.
$Q(t)$	Demand for capital investment.
$R(t)$	Market inflation index
μ	Sensitivity to price changes (market demand responsiveness)
η	Saving propensity.
λ	Expenditure on investment (demand for investment).

3. METHODOLOGICAL FRAMEWORK FOR MODELING THE NONLINEAR CHAOTIC STIFF FINANCIAL SYSTEM

The four cases for the NCFS model are formulated with the variations done in the parameter values and initial conditions of *P(t)*, *Q(t)*, and, *R(t)*. To explore the changes in NCFS model parameters i.e., saving propensity i.e., η, expenditure on investment i.e., λ, and sensitivity to price changes i.e., μ and initial conditions influence the behavior of the underlying system the cases are formulated and illustrated in Table 2. For case 1 the value of η is varied for four times and this gives the first case with variation 1, variation 2, variation 3, and variation 4 respectively. Similarly, the variation in the values of λ formulate the second case with its four variants. The third case is formulated by altering μ (sensitivity to price changes) values for four times. The final fourth case is formulated with the changes in the values of the initial conditions i.e., P (0), Q (0), and, R (0) of the NCFS model.

The mathematical depictions of the NCFS model related to the four cases and its variations i.e., case-1(variation-1), case-1(variation-3), case-1(variation-4), case-2 (variation-2), case-2 (variation-3), case-2 (variation-4), case-3 (variation-2), case-3 (variation-3), case-3 (variation-4), case-4 (variation-2), and case-4(variation-3) are presented in Table 3 for a clear and better view of the cases formulation and NCFS dynamics. The first column represents the NCFS model mathematical depictions while the initial conditions of P (0), Q (0), and, R (0) are given in column 2 of the Table 3.

Table 2 Formulation of the various cases and associated variations of NCFS model

Cases	μ	λ	η	P (0)	Q (0)	R (0)
Case-1(Variation-1)	1.0	0.10	2.36	2	3	2
Case-1(Variation-2)	1.0	0.10	3.0	2	3	2
Case-1(Variation-3)	1.0	0.10	0.9	2	3	2
Case-1(Variation-4)	1.0	0.10	4.3	2	3	2
Case-2(Variation-1)	1.0	0.10	3.0	2	3	2
Case-2(Variation-2)	1.0	0.2	3.0	2	3	2
Case-2(Variation-3)	1.0	0.19	3.0	2	3	2
Case-2(Variation-4)	1.0	0.13	3.0	2	3	2
Case-3(Variation-1)	1.0	0.10	3.0	2	3	2
Case-3(Variation-2)	2.0	0.10	3.0	2	3	2
Case-3(Variation-3)	1.45	0.10	3.0	2	3	2
Case-3(Variation-4)	1.3	0.10	3.0	2	3	2
Case-4(Variation-1)	1.0	0.10	0.9	2	3	2
Case-4(Variation-2)	1.0	0.10	0.9	1	2	0.5
Case-4(Variation-3)	1.0	0.10	0.9	0.1	0.2	0.6
Case-4(Variation-4)	1.0	0.10	0.9	0	2	0.1

Table 3 Mathematical depictions of the NCFS model

	NCFS Mathematical Depictions.	Initial conditions of state variables
Case-1(Variation-1)	$P'(t)=R(t)+P(t)(Q(t)-2.36),$ $Q'(t)=-P^2(t)+1-0.10Q(t),$ $R'(t)=-R(t)-P(t),$	$P(0)=2, Q(0)=3, R(0)=2$
Case-1(Variation-3)	$P'(t)=R(t)+P(t)(Q(t)-3.0),$ $Q'(t)=-P^2(t)+1-0.10Q(t),$ $R'(t)=-R(t)-P(t),$	$P(0)=2, Q(0)=3, R(0)=2$
Case-1(Variation-4)	$P'(t)=R(t)+P(t)(Q(t)-4.3),$ $Q'(t)=-P^2(t)+1-0.10Q(t),$ $R'(t)=-R(t)-P(t),$	$P(0)=2, Q(0)=3, R(0)=2$
Case-2(Variation-2)	$P'(t)=R(t)+P(t)(Q(t)-3.0),$ $Q'(t)=-P^2(t)+1-0.2Q(t),$ $R'(t)=-R(t)-P(t),$	$P(0)=2, Q(0)=3, R(0)=2$

Case-2(Variation-3)	$P'(t)=R(t)+P(t)(Q(t)-3.0),$ $Q'(t)=-P^2(t)+1-0.19Q(t),$ $R'(t)=-R(t)-P(t),$	$P(0)=2, Q(0)=3, R(0)=2$
Case-2(Variation-4)	$P'(t)=R(t)+P(t)(Q(t)-3.0),$ $Q'(t)=-P^2(t)+1-0.13Q(t),$ $R'(t)=-R(t)-P(t),$	$P(0)=2, Q(0)=3, R(0)=2$
Case-3(Variation-2)	$P'(t)=R(t)+P(t)(Q(t)-3.0),$ $Q'(t)=-P^2(t)+1-0.10Q(t),$ $R'(t)=-2R(t)-P(t),$	$P(0)=2, Q(0)=3, R(0)=2$
Case-3(Variation-3)	$P'(t)=R(t)+P(t)(Q(t)-3.0),$ $Q'(t)=-P^2(t)+1-0.10Q(t),$ $R'(t)=-1.45R(t)-P(t),$	$P(0)=2, Q(0)=3, R(0)=2$
Case-3(Variation-4)	$P'(t)=R(t)+P(t)(Q(t)-3.0),$ $Q'(t)=-P^2(t)+1-0.10Q(t),$ $R'(t)=-1.3R(t)-P(t),$	$P(0)=2, Q(0)=3, R(0)=2$
Case-4(Variation-2)	$P'(t)=R(t)+P(t)(Q(t)-0.9),$ $Q'(t)=-P^2(t)+1-0.10Q(t),$ $R'(t)=-R(t)-P(t),$	$P(0)=1, Q(0)=2, R(0)=0.5$
Case-4(Variation-3)	$P'(t)=R(t)+P(t)(Q(t)-0.9),$ $Q'(t)=-P^2(t)+1-0.10Q(t),$ $R'(t)=-R(t)-P(t),$	$P(0)=0.1, Q(0)=0.2, R(0)=0.6$

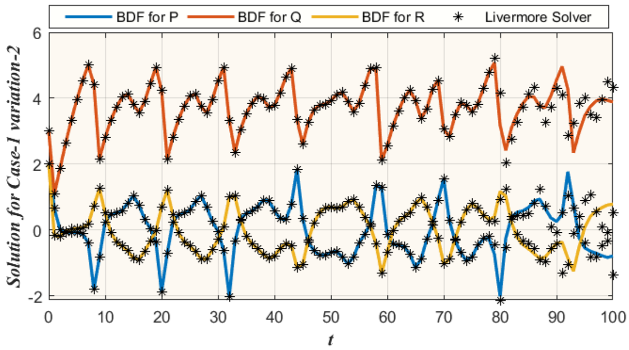
4. INTERPRETATION AND DISCUSSION OF THE RESULTS

In this section the comparative analysis based on the exploitation of various numerical solvers in generating the reference solutions of the NCFS model is presented with the aid of solution graphs and the evaluation related to the performance of the numerical solvers are illustrated via error analysis plots where AE metric has been used for evaluation criteria and the objective function is to achieve the minimum possible error, a desirable value of zero. The solution dynamics of NCFS model for the key variables loan interest rate *P(t)*, demand for capital investment *Q(t)* and market inflation index *R(t)* are presented in part a of the Figures 1, 2, 3, and 4, whereas part b of these Figures 1-4 represents the error analysis for four cases and 4 variations of NCFS model. In Figure 1, LSODE and BDF methods are exploited to solve the NCFS case 1 along with four variations, here the LSODE solutions are taken as the reference outcomes and compared with the BDF generated solutions. The solution dynamics shows the overlapping between the generated outcomes indicating the efficient, reliable solutions of the NCFS model for each of the variations of the case 1. The performance is evaluated on the basis of AE analysis that shows that for case 1(variation 2) the error is between 10^{-10} to 10^{-2} for variation 3 it also ranges from 10^{-10} to 10^{-2}, however for case 1: variation 1, and 4 the solutions of NCFS model are exactly similar to each other so the AE is zero so the graphs for case-1(variation-1, variation-2) have not been displayed here.

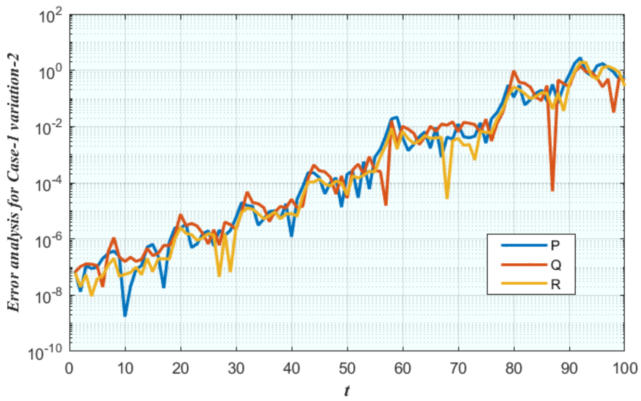
In Figure 2, Explicit and implicit Runge-Kutta (RKE, RKI) is being exploited to solve the case 2 and its variants of the NCFS model and in the second case the RKE is devoted as the benchmark for reference NCFS solutions and the RKI solutions are being compared with the benchmark solutions to evaluate the performance. The variation 1 of case 2 observes no difference in the NCFS solutions produced by the RKE and RKI and the error is zero here the actual solutions are identical to the generated outcomes by the RKI. However, comparing the solutions of NCFS with the employment of RKE and RKI are overlapping with negligible error, here for case 2 -variation 2 the value of error ranges from 10^{-10} to 10^{-2}, while for variation 3 the range is between 10^{-8} to 10^{-3} and finally for variation 4 the range is from 10^{-10} to 10^{-2}.for all of the key variable's *P(t)*, *Q(t)*, and, *R(t)* of NCFS model. In Figure 3 results of the case 3 with variation 1,2,3, and 4 are illustrated in the form of solution graph in part a and AE analysis in part b. The NCFS solutions are extracted with the aid of two numerical solvers i.e., BDF and RKI. In this case BDF outcomes for NCFS model are taken as the reference outcomes to compare with the RKI solutions. The comparative analysis shows the overlying of the solutions as the error between the reference and the predicted solutions by the RKI is approximately zero. As the error range is 10^{-10} to 10^{-2} for case 3(variation 1), 10^{-10} to 10^{-4} for case 3(variation 2), 10^{-9} to 10^{-4} for variation 3, and 10^{-9} to 10^{-3}.

The employment of the two numerical techniques namely

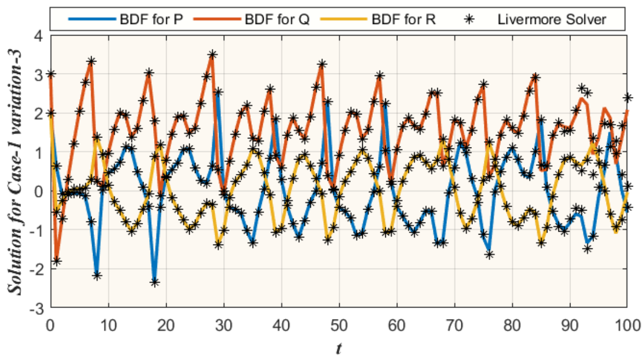
LSODE and RKE for solving the variations of case 4 of NCFS model is portrayed in Figure 4 with part a representing the solution dynamics and AE analysis for case 4 of NCFS model. In case 4, RKE solutions for NCFS model are considered as the reference solution, while comparing the reference solutions with the LSODE generated solutions for NCFS model, the difference seems to be negligible or it can be seen that the solutions are almost similar with little margin of error between them. The AE error almost ranges between 10-10 to 10-2 for all 4 variations of case 4 of NCFS model while solving with the RKE and LSODE numerical methods. This demonstrates the efficient performance of the numerical solvers being exploited for solving the diverse cases along with the associated variations in NCFS model parameters and initial conditions for robust analysis.



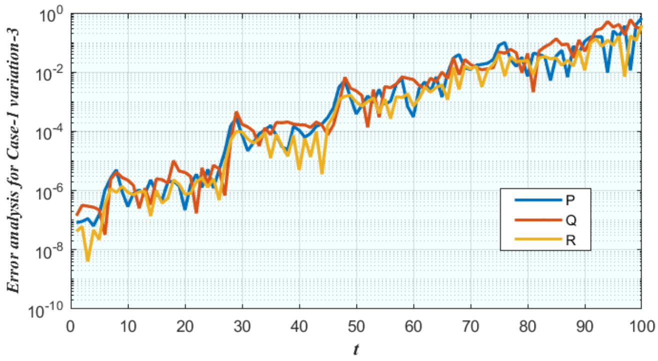
(a) Dynamics for case-1(variation-2)



(b) error analysis



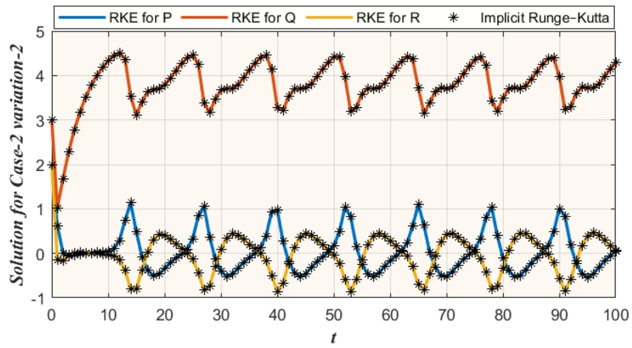
(a) Dynamics for case-1(variation-3)



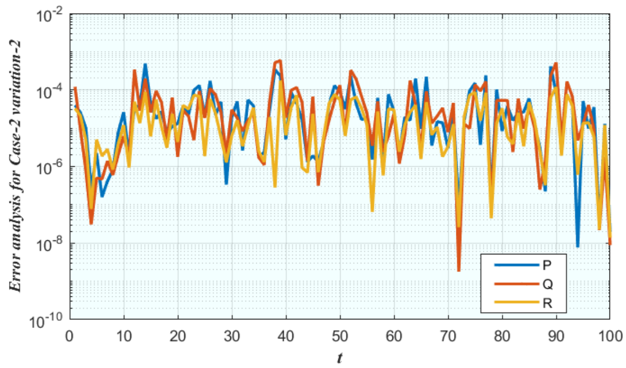
(b) error analysis

Fig. 1 Dynamics for NCFS case-1 (variation 1, variation 2, and variation 3).

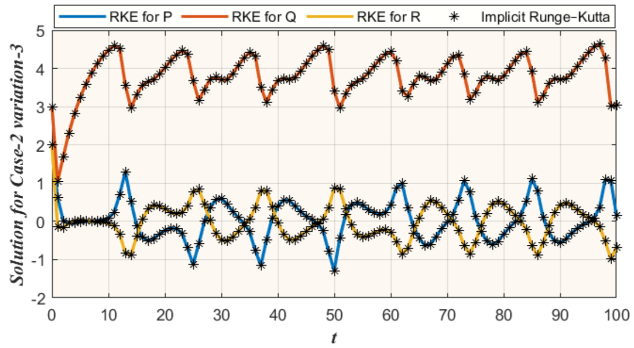
Note: LSODE and BDF methods are exploited to solve the NCFS case 1 along with four variations, here the LSODE solutions are taken as the reference outcomes and compared with the BDF generated solutions.



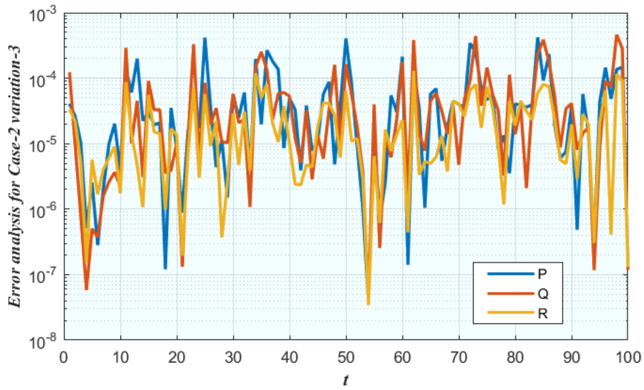
(a) Dynamics for case-2(variation-2)



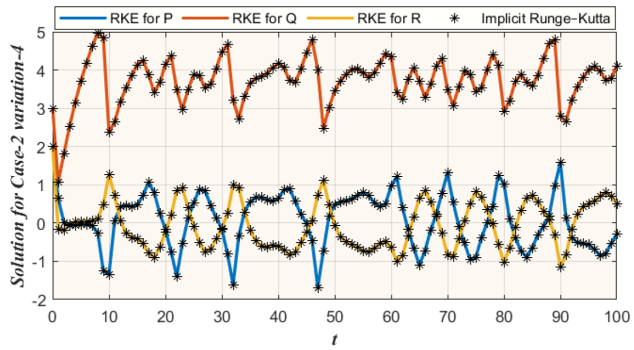
(b) error analysis



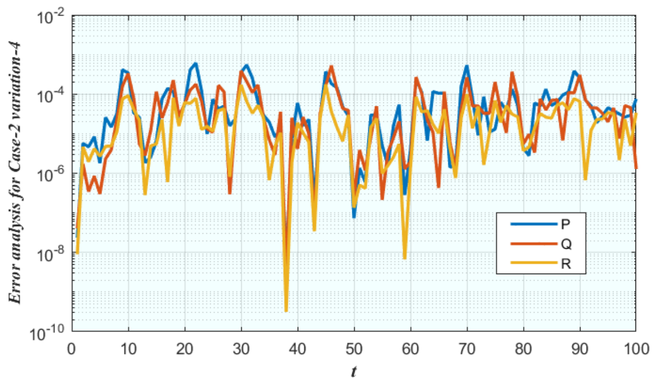
(a) Dynamics for case-2(variation-3)



(b) error analysis



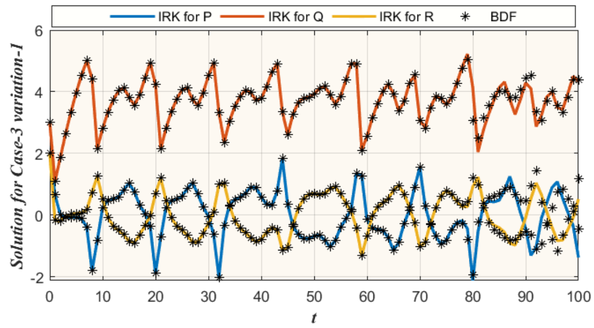
(a) Dynamics f



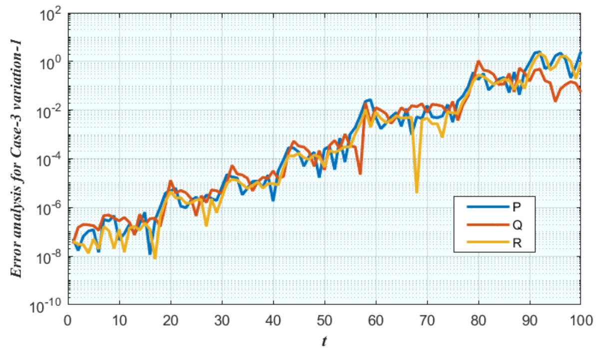
(b) error analysis

Fig. 2 Dynamics for NCFS case-2 (variation 1, variation 2, and variation 3).

Note: Explicit and implicit Runge-Kutta (RKE, RKI) is being exploited to solve the case 2 and its variants of the NCFS model and in the second case the RKE is devoted as the benchmark for reference NCFS solutions and the RKI solutions are being compared with the benchmark solutions to evaluate the performance.



(a) Dynamics for case-3(variation-1)



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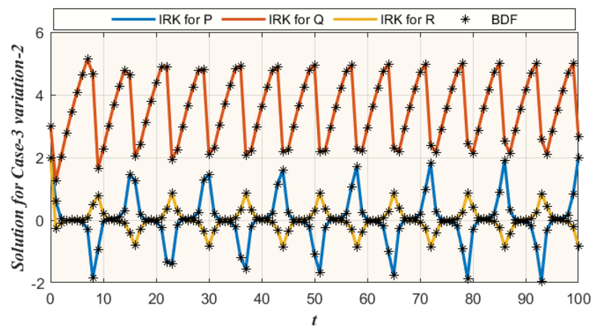
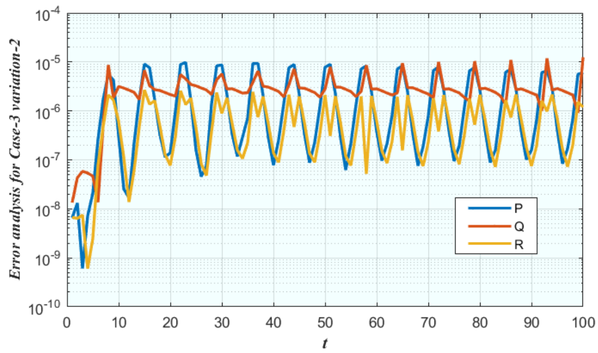


(a) Dynamics for case-3(variation-2)



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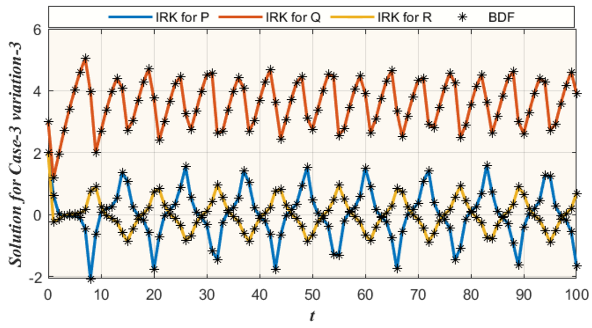
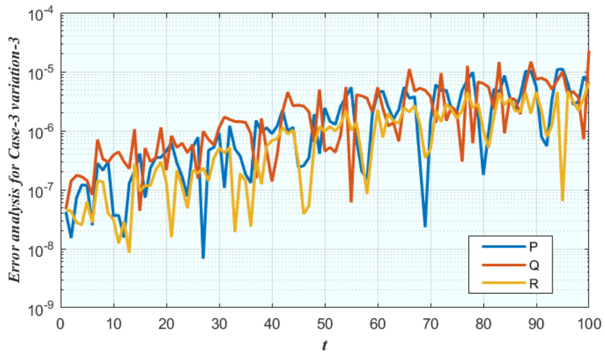


(a) Dynamics for case-3(variation-3)



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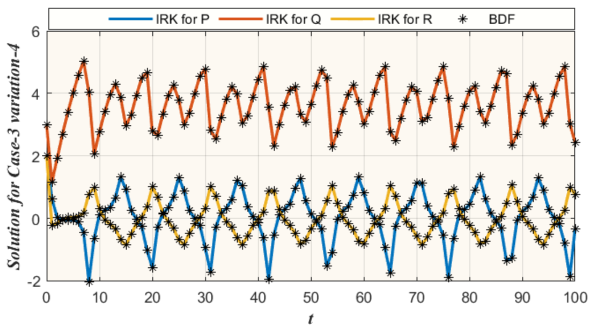
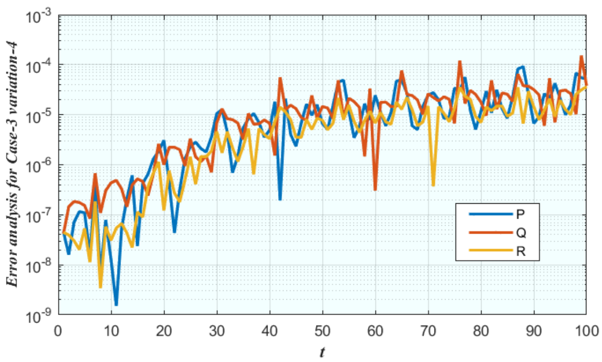


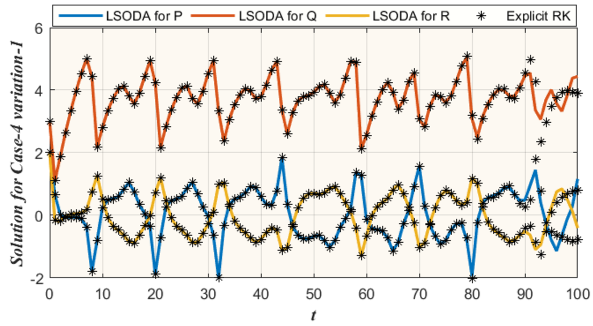
(a) Dynamics for case-3(variation-4)



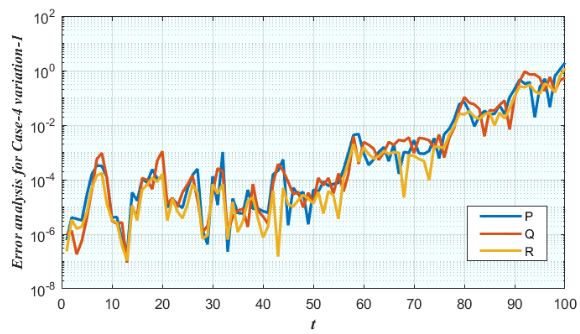
(b) error analysis

Fig. 3 Dynamics for NCFS case-3 (variation 1, variation 2, and variation 3).

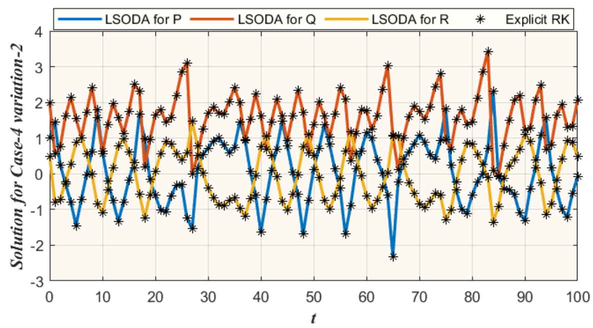
Note: The NCFS solutions are extracted with the aid of two numerical solvers i.e., BDF and RKI. In this case BDF outcomes for NCFS model are taken as the reference outcomes to compare with the RKI solutions.



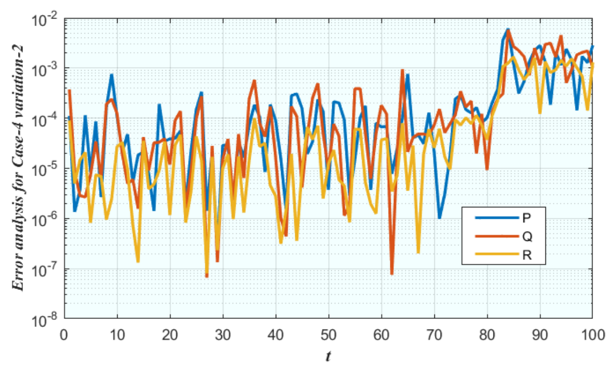
(a) Dynamics for case-4(variation-1)



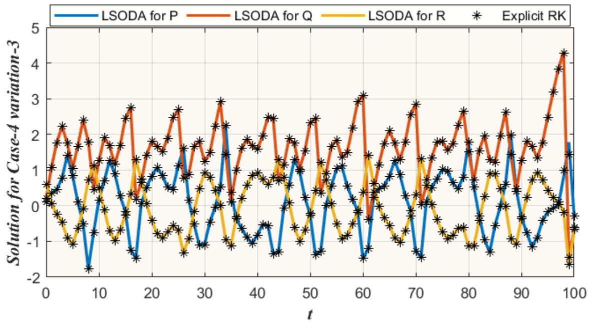
(b) error analysis



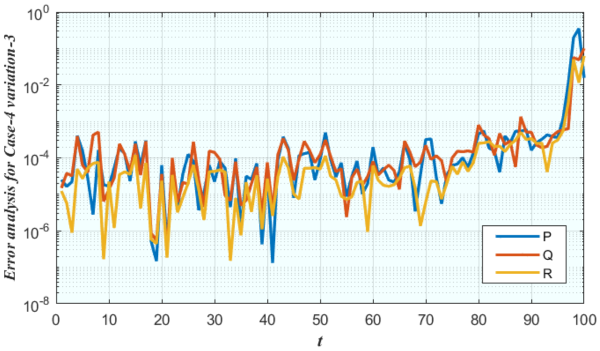
(a) Dynamics for case-4(variation-2)



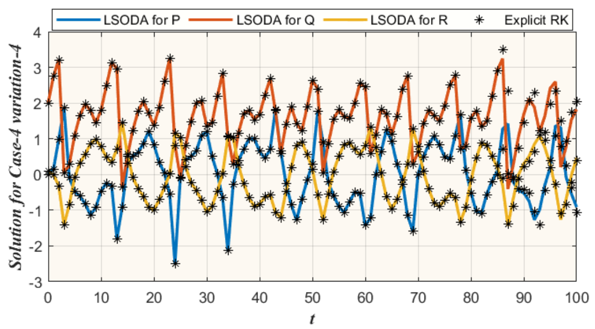
(b) error analysis



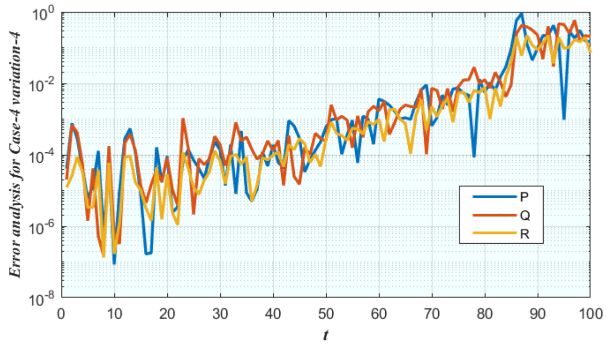
(a) Dynamics for case-4(variation-3)



(b) error analysis



(a) Dynamics for case-4(variation-4)



(b) error analysis

Fig. 4 Dynamics for NCFS case-4 (variation 1, variation 2, and variation 3).

Note: RKE solutions for NCFS model are considered as the reference solution, while comparing the reference solutions with the LSODE generated solutions for NCFS model.

5. CONCLUSION

The presented strategy explores the dynamics of the NCFS modelled with the set of ODEs that comprises the key variables in the form of loan interest rate, demand for capital investment and market inflation index. By systematically varying the parameter values including saving propensity, expenditure on investment, and sensitivity to price changes i.e., market demand responsiveness and starting values of the NCFS state variables results in the formulation of diverse case studies along with the different variants. This offers a comprehensive analysis of the system behavior under varying conditions and allows for the examination of the system sensitivity as how the NCFS model responds to these changes. Through the aid of numerical simulations, the analysis based on NCFS model dynamics is carried out with the exploitation of various numerical solvers including Livermore solver, backward differentiation formula (BDF), explicit Runge-Kutta and implicit Runge-Kutta methods. The numerical solvers employed in this study for solving the complex financial system, simulate the accurate, efficient and reliable dynamics of the NCFS model. The comprehensive analysis based on the comparative evaluation of the employed numerical methods indicates the strength and efficiency in generating the reliable synthetic data for NCFS model that depicts the real-world scenario of the financial system under changing conditions. The error analysis based on the AE metrics utilized to evaluate the performance of the numerical solvers in generating the reference solutions for the NCFS model. The AE value is minimum and as low as 10-10 with the employment of the numerical methods for all the concerning cases and variations formulated for the NCFS model. The findings underscore that each of the numerical solvers are capable of diverse strengths and approximate ODE's solutions to facilitate researchers in exploring complex and dynamic interactions in diversified fields of finance, social sciences, business, engineering, and economics.

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